On symplectic folding

Felix Schlenk

10. January 1999

Abstract

We study the rigidity and flexibility of symplectic embeddings of simple shapes. It is first proved that under the condition $r_n^2 \leq 2r_1^2$ the symplectic ellipsoid $E(r_1, \dots, r_n)$ with radii $r_1 \leq \dots \leq r_n$ does not embed in a ball of radius strictly smaller than r_n. We then use symplectic folding to see that this condition is sharp and to construct some nearly optimal embeddings of ellipsoids and polydiscs into balls and cubes. It is finally shown that any connected symplectic manifold of finite volume may be asymptotically filled with skinny ellipsoids or polydiscs.

Contents

1 Introduction

Let U be an open subset of $\mathbb{R}^n$ which is diffeomorphic to a ball, endow U with the Euclidean volume form Ω_0, and let (M, Ω) be any connected n-dimensional volume manifold. Then U embeds into M via a volume preserving map if and only if $\mathrm{Vol}\,(U, \Omega_0) \le \mathrm{Vol}\,(M, \Omega)$. (A proof of this "folk-theorem" is given below.)

Let $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ be the standard symplectic form on $\mathbb{R}^{2n}$ and equip any open subset U of $\mathbb{R}^{2n}$ with this form. An embedding $\varphi\colon U \hookrightarrow \mathbb{R}^{2n}$ is called symplectic, if $\varphi^*\omega_0 = \omega_0$. In particular, every symplectic embedding preserves the volume and the orientation. In dimension two, the converse holds true. In higher dimensions, however, strong symplectic rigidity phenomena appear. A spectacular example for this is Gromov's Nonsqueezing Theorem [12], which states that a ball $B^{2n}(r)$ of radius r symplectically embeds in the standard symplectic cylinder $B^2(R) \times \mathbb{R}^{2n-2}$ if and only if $r \le R$. This and many other rigidity results for symplectic maps could later be explained via symplectic capacities which arose from the variational study of periodic orbits of Hamiltonian systems (see [14] and the references therein).

On the other hand, the flexibility of symplectic codimension 2 embeddings of open manifolds [13, p. 335] implies that given any symplectic ball B^{2n-2} in $\mathbb{R}^{2n-2}$ and a symplectic manifold (M^{2n}, ω), there exists an $\epsilon > 0$ such that $B^{2n-2} \times B^2(\epsilon)$ symplectically embeds in M (see [10, p. 579] for details).

The aim of this work is to investigate the zone of transition between rigidity and flexibility in symplectic topology. Unfortunately, symplectic capacities can be computed only for very special sets, and there is still not much known about what one can do with a symplectic map. We thus look at a model situation. Let

$$E(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^n \frac{\pi |z_i|^2}{a_i} < 1 \right\}$$

be the open symplectic ellipsoid with radii $\sqrt{a_i/\pi}$, and write $D(a)$ for the open disc of area a and $P(a_1, \dots, a_n)$ for the polydisc $D(a_1) \times \dots \times D(a_n)$. Since a permutation of the symplectic coordinate planes is a (linear) symplectic map, we may assume $a_i \leq a_j$ for $i < j$. Finally, denote the ball $E^{2n}(a, \dots, a)$ by $B^{2n}(a)$ and the "n-cube" $P^{2n}(a, \dots, a)$ by $C^{2n}(a)$. We call any of these sets a simple shape. We ask:

"Given a simple shape S, what is the smallest ball B and what is the smallest cube C such that S symplectically fits into B and C?"

Observe that embedding S into a minimal ball amounts to minimizing its diameter, while embedding S into a minimal cube amounts to minimizing its symplectic width.

Our main rigidity result states that for "round" ellipsoids the identity provides already the optimal embedding.

Theorem 1 *Let $a_n \leq 2a_1$ and $a < a_n$. Then $E(a_1, \dots, a_n)$ does not embed symplectically in $B^{2n}(a)$.*

An ordinary symplectic capacity only shows that if $a < a_1$, there is no symplectic embedding of $E(a_1, \dots, a_n)$ into $B^{2n}(a)$. Our proof uses the first n Ekeland-Hofer capacities. For $n = 2$, Theorem 1 was proved in [10] as an early application of symplectic homology, but the argument given here is much simpler and works in all dimensions.

Our first flexibility result states that Theorem 1 is sharp.

Theorem 2A *Given any $\epsilon > 0$ and $a > 2\pi$, there exists a symplectic embedding*

$$E^{2n}(\pi, \dots, \pi, a) \hookrightarrow B^{2n}\left(\frac{a}{2} + \pi + \epsilon\right).$$

Lalonde and McDuff observed in [18] that their technique of symplectic folding can be used to prove Theorem 2A for $n = 2$. The symplectic folding construction considers a 4-ellipsoid as a fibration of discs of varying size over a disc and applies the flexibility of volume preserving maps to both the base and the fibres. It is therefore purely four dimensional in nature. We refine the method in such a way that it will nevertheless be sufficient to prove the result for arbitrary dimension.

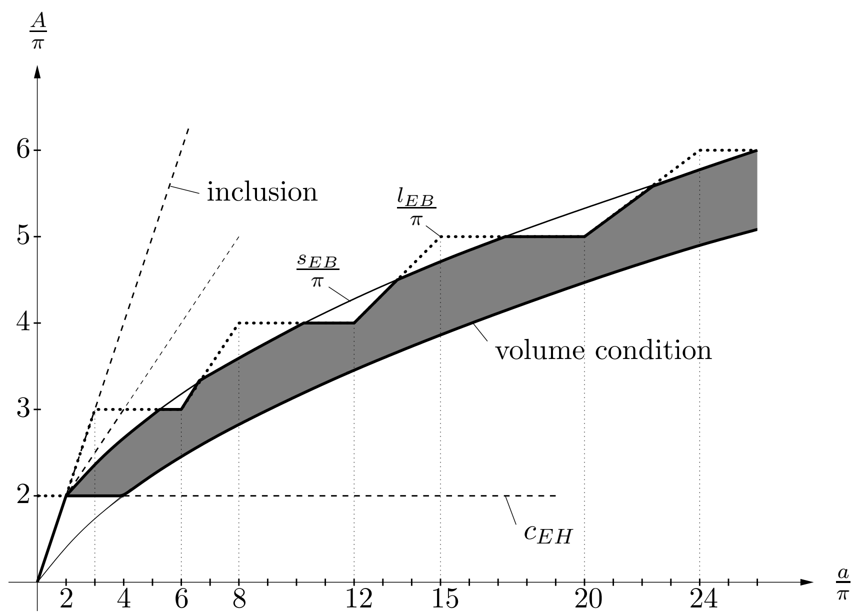

Figure 1: What is known about $E(\pi, a) \hookrightarrow B^4(A)$

Theorem 1 and Theorem 2A shed some light on the power of Ekeland-Hofer capacities: As soon as these invariants cease to imply that there is no better embedding than the identity, there is indeed a better embedding.

For embeddings of ellipsoids into cubes, the same procedure yields a similarly sharp result, but for embeddings of polydiscs into balls and cubes the result is less satisfactory. In four dimensions, the precise result is as follows.

Theorem 2B *Let ϵ be any positive number.*

(i) Let $a > \pi$. Then there is no symplectic embedding of $E(\pi, a)$ into $C^4(\pi)$, but $E(\pi, a)$ symplectically embeds in $C^4(\frac{a+\pi}{2} + \epsilon)$.

(ii) Let $a > 2\pi$. Then $P(\pi, a)$ symplectically embeds in $B^4(\frac{a}{2} + 2\pi + \epsilon)$ as well as in $C^4(\frac{a}{2} + \pi + \epsilon)$.

Question 1 *Does $P(\pi, 2\pi)$ symplectically embed in $B^4(A)$ for some $A < 3\pi$ or in $C^4(A)$ for some $A < 2\pi$?*

Both, Theorem 2A and Theorem 2B as well as its higher dimensional version can be substantially improved by multiple folding. Let us discuss the result

in case of embeddings of 4-ellipsoids into 4-balls (cf. Figure 1). Let $s_{EB}(a)$ be the function describing the best embeddings obtainable by symplectic folding. It turns out that

$$\limsup_{\epsilon \to 0^+} \frac{s_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} = \frac{3}{7}.$$

Question 2 *Let $f_{EB}(a) = \inf\{A \,|\, E(\pi, a)$ symplectically embeds in $B^4(A)\}$. How does f_{EB} look like near 2π? In particular,*

$$\limsup_{\epsilon \to 0^+} \frac{f_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} < \frac{3}{7}\,?$$

Moreover, as $a \to \infty$ the image of $E(\pi, a)$ fills up an arbitrarily large percentage of the volume of $B^4(s_{EB}(a))$. This can also be seen via a Lagrangian folding method, which was developed by Traynor in [31] and yielded the best previously known results for the above embedding problem (see the curve l_{EB} in Figure 1). Symplectic folding, however, may be used to prove that any connected symplectic manifold (M, ω) of finite volume can be asymptotically filled by skinny ellipsoids and polydiscs: For $a > \pi$ set

$$p_a^E(M^{2n}, \omega) = \sup_{\alpha} \frac{\operatorname{Vol}(E^{2n}(\alpha\pi, \dots, \alpha\pi, \alpha a))}{\operatorname{Vol}(M, \omega)},$$

where the supremum is taken over all α for which $E^{2n}(\alpha\pi, \dots, \alpha\pi, \alpha a)$ symplectically embeds in (M, ω), and define $p_a^P(M, \omega)$ in a similar way.

Theorem 3 $\lim_{a \to \infty} p_a^E(M, \omega)$ *and* $\lim_{a \to \infty} p_a^P(M, \omega)$ *exist and equal 1.*

This result exhibits that in the limit symplectic rigidity disappears. We finally give estimates of the convergence speed from below.

Appendix A provides computer programs necessary to compute the optimal embeddings of ellipsoids into a 4-ball and a 4-cube obtainable by our methods, and in Appendix B we give an overview on known results on the Gromov width of closed symplectic manifolds.

Acknowledgement. I greatly thank Dusa McDuff for her fine criticism on an earlier more complicated attempt towards Theorem 2A, which gave worse estimates, and for having explained to me the main point of the folding construction.

I also thank Leonid Polterovich for suggesting to me to look closer at Lagrangian folding.

2 Rigidity

Throughout this paper, if there is no explicit mention to the contrary, all maps will be assumed to be symplectic. In dimension two this just means that they preserve the orientation and the area.

Denote by $\mathcal{O}(n)$ the set of bounded domains in $\mathbb{R}^{2n}$ endowed with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Given $U \in \mathcal{O}(n)$, write $|U|$ for the volume of U with respect to the Euclidean volume form $\Omega_0 = \frac{1}{n!}\omega_0^n$. Let $\mathcal{D}(n)$ be the group of symplectomorphisms of $\mathbb{R}^{2n}$ and $\mathcal{D}_c(n)$ respectively $Sp(n;\mathbb{R})$ the subgroups of compactly supported respectively linear symplectomorphisms of $\mathbb{R}^{2n}$. Define the following relations on $\mathcal{O}(n)$:

$$\begin{array}{lcl} U \le_1 V & \iff & \text{There exists a } \varphi \in Sp(n;\mathbb{R}) \text{ with } \varphi(U) \subset V. \\ U \le_2 V & \iff & \text{There exists a } \varphi \in \mathcal{D}(n) \text{ with } \varphi(U) \subset V. \\ U \le_3 V & \iff & \text{There exists a symplectic embedding } \varphi\colon U \hookrightarrow V. \end{array}$$

Of course, $\le_1 \Rightarrow \le_2 \Rightarrow \le_3$, but all the relations are different: That $\le_1$ and $\le_2$ are different is well known (see (2) below and Traynor's theorem stated at the beginning of section 3). The construction of sets U and $V \in \mathcal{O}(n)$ with $U \le_3 V$ but $U \not\le_2 V$ relies on the following simple observation. Suppose that U and V not only fulfill $U \le_3 V$ but are symplectomorphic, whence, in particular, $|U| = |V|$. Thus, if $U \le_2 V$ and φ is a map realizing $U \le_2 V$, no point of $\mathbb{C}^n \setminus U$ can be mapped to V, and we conclude that $\varphi(\partial U) = \partial V$. In particular, the characteristic foliations on ∂U and ∂V are isomorphic, and if ∂U is of contact type, then so is ∂V (see [14] for basic notions in Hamiltonian dynamics).

Let now $U = B^{2n}(\pi)$, let

$$SD = D(\pi) \setminus \{(x,y) \mid x \ge 0, y = 0\}$$

be the slit disc and set $V = B^{2n}(\pi) \cap (SD \times \cdots \times SD)$. Traynor proved in [31] that for $n \le 2$, V is symplectomorphic to $B^{2n}(\pi)$. But ∂U and ∂V are not even diffeomorphic. For $n \ge 2$ very different examples were found in [8] and [4]. Theorem 1.1 in [8] and its proof show that there exist $U, V \in \mathcal{O}(n)$ with smooth convex boundaries such that U and V are symplectomorphic

and C^∞-close to $B^{2n}(\pi)$, but the characteristic foliation of ∂U contains an isolated closed orbit while the one of ∂V does not. And Corollary A in [4] and its proof imply that given any $U \in \mathcal{O}(n)$, $n \geq 2$, with smooth boundary ∂U of contact type, there exists a symplectomorphic and C^0-close $V \in \mathcal{O}(n)$ whose boundary is not of contact type.

We in particular see that even for U being a ball, $\leq_3$ does not imply $\leq_2$.

In order to detect some rigidity via the above relations we therefore must pass to a small subcategory of sets:

Let $\mathcal{E}(n)$ be the collection of symplectic ellipsoids described in the introduction

$$\mathcal{E}(n) = \{E(a) = E(a_1, \dots, a_n)\}$$

and write $\preccurlyeq_i$ for the restrictions of the relations $\leq_i$ to $\mathcal{E}(n)$.

Notice again that

$$\preccurlyeq_1 \implies \preccurlyeq_2 \implies \preccurlyeq_3 \; .$$

$\preccurlyeq_2$ and $\preccurlyeq_3$ are actually very similar: Since ellipsoids are starlike, we may apply Alexander's trick to prove the extension after restriction principle (see [6] for details), which tells us that given any embedding $\varphi\colon E(a) \hookrightarrow E(a')$ and any $\delta \in]0,1[$ we can find a $\psi \in \mathcal{D}(n)$ which coincides with φ on $E(\delta a)$; hence

$$E(a) \preccurlyeq_3 E(a') \implies E(\delta a) \preccurlyeq_2 E(a') \quad \text{for all } \delta \in]0,1[\, . \tag{1}$$

It is, however, not clear whether $\preccurlyeq_2$ and $\preccurlyeq_3$ are the same: While Theorem 2.2 proves this under an additional condition, the folding construction of section 3 suggests that they are different in general. But let us first prove a general and common rigidity property of these relations:

Proposition 2.1 *The relations $\preccurlyeq_i$ are partial orderings on $\mathcal{E}(n)$.*

Proof. The relations are clearly reflexive and transitive, so we are left with identitivity. Of course, the identitivity of $\preccurlyeq_3$ implies the one of $\preccurlyeq_2$ which, in its turn, implies the one of $\preccurlyeq_1$. We still prefer to give independent proofs which use tools whose difficulty is about proportional to the depth of the results.

It is well known from linear symplectic algebra [14, p. 40] that

$$E(a) \preccurlyeq_1 E(a') \iff a_i \leq a'_i \quad \text{for all } i, \tag{2}$$

in particular $\preccurlyeq_1$ is identitive.

Given $U \in \mathcal{O}(n)$ with smooth boundary ∂U, the spectrum $\sigma(U)$ of U is defined to be the collection of the actions of closed characteristics on ∂U. It is clearly invariant under $\mathcal{D}(n)$, and for an ellipsoid it is given by

$$\sigma(E(a_1,\dots,a_n)) = \{d_1(E) \le d_2(E) \le \dots\} \stackrel{\text{def}}{=} \{ka_i \mid k \in \mathbb{N},\, 1 \le i \le n\}.$$

Let now φ be a map realizing $E(a) \preccurlyeq_2 E(a')$. $E(a) \preccurlyeq_2 E(a') \preccurlyeq_2 E(a)$ gives in particular $|E(a)| = |E(a')|$, and we conclude as above that $\varphi(\partial E(a)) = \partial E(a')$. This implies $\sigma(E(a)) = \sigma(E(a'))$ and the claim for $\preccurlyeq_2$ follows.

To prove identitivity of $\preccurlyeq_3$ recall that Ekeland-Hofer capacities [7] provide us with a whole family of symplectic capacities for subsets of $\mathbb{C}^n$. They are invariant under $\mathcal{D}_c(n)$, and for an ellipsoid E they are given by the spectrum:

$$\{c_1(E) \le c_2(E) \le \dots\} = \{d_1(E) \le d_2(E) \le \dots\}. \tag{3}$$

First observe that in the proof of the extension after restriction principle the generating Hamiltonian can be chosen to vanish outside a large ball, so the extension can be assumed to be in $\mathcal{D}_c(n)$. This shows that in the definition of $\preccurlyeq_2$ we may replace $\mathcal{D}(n)$ by $\mathcal{D}_c(n)$ without changing the relation, and that Ekeland-Hofer capacities may be applied to $\preccurlyeq_2$. Next observe that for any $i \in \{1,2,3\}$ and $\alpha > 0$

$$E(a) \preccurlyeq_i E(a') \implies E(\alpha a) \preccurlyeq_i E(\alpha a'), \tag{4}$$

just conjugate the given map φ with the dilatation by α^{-1}. Applying this and (1) we see that for any $\delta_1, \delta_2 \in]0,1[$ the assumed relations

$$E(a) \preccurlyeq_3 E(a') \preccurlyeq_3 E(a)$$

imply

$$E(\delta_2\delta_1 a) \preccurlyeq_2 E(\delta_1 a') \preccurlyeq_2 E(a),$$

and now the monotonicity of all the $c_i = d_i$ immediately gives $a = a'$. $\Box$

It is well known (we refer again to the beginning of section 3) that $\preccurlyeq_2$ does not imply $\preccurlyeq_1$ in general. However, a suitable pinching condition guarantees that "linear" and "non linear" coincide:

Theorem 2.2 *Let $\kappa \in]\frac{\pi}{2}, \pi[$. Then the following statements are equivalent:*

(i) $B^{2n}(\kappa) \preccurlyeq_1 E(a) \preccurlyeq_1 E(a') \preccurlyeq_1 B^{2n}(\pi)$

(ii) $B^{2n}(\kappa) \preccurlyeq_2 E(a) \preccurlyeq_2 E(a') \preccurlyeq_2 B^{2n}(\pi)$

(iii) $B^{2n}(\kappa) \preccurlyeq_3 E(a) \preccurlyeq_3 E(a') \preccurlyeq_3 B^{2n}(\pi)\,.$

Theorem 1 follows from Theorem 2.2, (2) and (4). For $n = 2$, Theorem 2.2 was proved in [10]. That proof uses a deep result by McDuff, namely that the space of symplectic embeddings of a ball into a larger ball is unknotted, and then applies the isotopy invariance of symplectic homology. However, Ekeland-Hofer capacities provide an easy proof. The crucial point is that as true capacities they have - very much in contrast to symplectic homology - the monotonicity property.

Proof of Theorem 2.2. (ii) $\Rightarrow$ (i): By assumption we have $B^{2n}(\kappa) \preccurlyeq_2 E(a) \preccurlyeq_2 B^{2n}(\pi)$, so the first Ekeland-Hofer capacity c_1 gives

$$\kappa \le a_1 \le \pi \tag{5}$$

and c_n gives

$$\kappa \le c_n(E(a)) \le \pi. \tag{6}$$

(5) and $\kappa > \frac{\pi}{2}$ imply $2a_1 > \pi$, whence the only elements in $\sigma(E(a))$ possibly smaller than π are $a_1, \dots, a_n$. It follows therefore from (6) that $a_n = c_n(E(a))$, whence $c_i(E(a)) = a_i$ $(1 \le i \le n)$. Similarly we find $c_i(E(a')) = a_i'$ $(1 \le i \le n)$, and from $E(a) \preccurlyeq_2 E(a')$ we conclude $a_i \le a_i'$.

(iii) $\Rightarrow$ (i) follows now by a similar reasoning as in the proof of the identitivity of $\preccurlyeq_3$: Starting from

$$B^{2n}(\kappa) \preccurlyeq_3 E(a) \preccurlyeq_3 E(a') \preccurlyeq_3 B^{2n}(\pi),$$

(1) shows that for any $\delta_1, \delta_2, \delta_3 \in]0,1[$

$$B^{2n}(\delta_3\delta_2\delta_1\kappa) \preccurlyeq_2 E(\delta_2\delta_1 a) \preccurlyeq_2 E(\delta_1 a') \preccurlyeq_2 B^{2n}(\pi)\,.$$

Choosing $\delta_1, \delta_2, \delta_3$ so large that $\delta_3\delta_2\delta_1\kappa > \frac{\pi}{2}$ we may apply the already proved implication to see

$$B^{2n}(\delta_3\delta_2\delta_1\kappa) \preccurlyeq_1 E(\delta_2\delta_1 a) \preccurlyeq_1 E(\delta_1 a) \preccurlyeq_1 B^{2n}(\pi),$$

and since $\delta_1, \delta_2, \delta_3$ may be chosen arbitrarily close to 1, (2) shows that we are done. $\square$

3 Flexibility

As it was pointed out in the introduction, the flexibility of symplectic codimension 2 embeddings of open manifolds implies that a condition as in Theorem 1 is necessary for rigidity. An explicit necessary condition was first obtained by Traynor in [31]. Her construction may be extended in an obvious way (see subsection 3.4, in particular Corollary 3.18 $(i)_E$) to prove

Theorem (Traynor, [31, Theorem 6.4]) *For all $k \in \mathbb{N}$ and $\epsilon > 0$ there exists a symplectic embedding*

$$E\left(\frac{\pi}{k+1}, \pi, \dots, \pi, k\pi\right) \hookrightarrow B^{2n}(\pi + \epsilon).$$

However, neither this theorem nor any refined version yielded by the Lagrangian method used in its proof can decide whether Theorem 1 is sharp (cf. Figure 1). Our first flexibility result states that this is indeed the case:

Theorem 3.1 *Let $a > 2\pi$ and $\epsilon > 0$. Then $E^{2n}(\pi, \dots, \pi, a)$ embeds symplectically in $B^{2n}(\frac{a}{2} + \pi + \epsilon)$.*

For $n = 2$, this theorem together with Theorem 1 gives a complete answer to our question in the introduction, whereas for arbitrary n it only states that Theorem 1 is sharp. We indeed cannot expect a much better result since (as is seen using Ekeland-Hofer capacities) $E^{2n}(\pi, 3\pi, \dots, 3\pi)$ does not embed in any ball of capacity strictly smaller than 3π.

Proof of Theorem 3.1. We will construct an embedding

$$\Phi \colon E(a, \pi) \hookrightarrow B^4\left(\frac{a}{2} + \pi + \epsilon\right)$$

satisfying

$$\pi|\Phi(z_1, z_2)|^2 < \frac{a}{2} + \epsilon + \frac{\pi^2|z_1|^2}{a} + \pi|z_2|^2 \quad \text{for all } (z_1, z_2) \in E(a, \pi). \tag{7}$$

The composition of the linear symplectomorphism

$$E^{2n}(\pi, \dots, \pi, a) \to E^{2n}(a, \pi, \dots, \pi)$$

with the restriction of $\Phi \times id_{2n-4}$ to $E^{2n}(a, \pi, \dots, \pi)$ is then the desired embedding.

The great flexibility of 2-dimensional area preserving maps is basic for the construction of Φ. We now make sure that we may describe such a map by prescribing it on an exhausting and nested family of loops.

Definition A family $\mathcal{L}$ of loops in a simply connected domain $U \subset \mathbb{R}^2$ is called *admissible* if there is a diffeomorphism $\beta \colon D(|U|) \setminus \{0\} \to U \setminus \{p\}$ for some point $p \in U$ such that

(i) concentric circles are mapped to elements of $\mathcal{L}$

(ii) in a neighbourhood of the origin β is an orientation preserving isometry.

Lemma 3.2 *Let U and V be bounded and simply connected domains in $\mathbb{R}^2$ of equal area and let $\mathcal{L}_U$ respectively $\mathcal{L}_V$ be admissible families of loops in U respectively V. Then there is a symplectomorphism between U and V mapping loops to loops.*

Remark. The regularity condition (ii) imposed on the families taken into consideration can be weakened. Some condition, however, is necessary as is seen from taking $\mathcal{L}_U$ a family of concentric circles and $\mathcal{L}_V$ a family of rectangles with round corners and width larger than a positive constant. $\diamond$

Proof of Lemma 3.2. We may assume that $(U, \mathcal{L}_U) = (D(\pi R^2), \{re^{i\phi}\})$, and after reparametrizing the r-variable by a diffeomorphism of $]0, R[$ which is the identity near 0 we may assume that β maps the loop $C(r)$ of radius r to the loop $L(r)$ in $\mathcal{L}_V$ which encloses the area πr^2.

We now search for a family $h(r, \cdot)$ of diffeomorphisms of S^1 such that the map α given by $\alpha(re^{i\phi}) = \beta(re^{ih(r,\phi)})$ is a symplectomorphism. With other words, we look for a smooth $h \colon]0, R[\times S^1 \to S^1$ which is a diffeomorphism for r fixed and solves the initial value problem

$$(*) \qquad \left\{ \begin{array}{lcl} \frac{\partial h}{\partial \phi}(r, \phi) & = & 1/\det \beta'(re^{ih(r,\phi)}) \\ h(r, 0) & = & 0 \end{array} \right.$$

View ϕ for a moment as a real variable. The existence and uniqueness theorem for ordinary differential equations with parameter yields a smooth map $h \colon]0, R[\times \mathbb{R} \to \mathbb{R}$ satisfying $(*)$. Thus, $h(r, \cdot)$ is a diffeomorphism of $\mathbb{R}$, and it remains to check that it is 2π-periodic. But this holds since the map $\alpha \colon re^{i\phi} \mapsto \beta(re^{ih(r,\phi)})$ locally preserves the volume and $\alpha(C(r))$ is contained in the loop $L(r)$.

Finally, α is an isometry in a punctured neighbourhood of the origin and thus extends to all of $D(\pi R^2)$. $\Box$

While Traynor's construction relies mainly on considering a 4-ellipsoid as a Lagrangian product of a rectangle and a triangle, we view it as a trivial fibration over a symplectic disc with symplectic discs of varying size as fibres: More generally, define for $U \subset \mathbb{C}$ open and $f\colon U \to \mathbb{R}_{>0}$

$$\mathcal{F}(U,f) = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1 \in U,\, \pi|z_2|^2 < f(z_1)\}.$$

This is the trivial fibration over U with fiber over z_1 the disc of capacity $f(z_1)$. For $\lambda \in \mathbb{R}$ set

$$U_\lambda = \{z_1 \in U \mid f(z_1) \geq \lambda\}.$$

Given two such fibrations $\mathcal{F}(U,f)$ and $\mathcal{F}(V,g)$, an embedding $\psi\colon U \hookrightarrow V$ defines an embedding $\psi \times id\colon \mathcal{F}(U,f) \hookrightarrow \mathcal{F}(V,g)$ if and only if $f(z_1) \leq g(\psi(z_1))$ for all $z_1 \in U$, and under the assumption that all the sets U_λ and V_λ are connected, we see from Lemma 3.2 that inequalities

$$\text{area } U_\lambda < \text{ area } V_\lambda \quad \text{ for all } \lambda$$

are sufficient for the existence of an embedding $\mathcal{F}(U,f) \hookrightarrow \mathcal{F}(V,g)$.

Example ([19, p. 54]) Let $T(a,b) = \mathcal{F}(R(a),g)$ with

$$R(a) = \{z_1 = (u,v) \mid 0 < u < a,\, 0 < v < 1\}$$

and $g(z_1) = g(u) = b - u$ be the trapezoid. We think of $T(a,b)$ as depicted in Figure 2. $\diamond$

Lemma 3.3 *For all $\epsilon > 0$,*

(i) $E(a,b)$ *embeds in* $T(a+\epsilon,b)$

(ii) $T(a,b)$ *embeds in* $E(a+\epsilon,b)$.

Proof. $E(a,b)$ is described by $U = D(a)$ and $f(z_1) = b\,(1 - \frac{\pi|z_1|^2}{a})$. For (i) look at α and for (ii) at ω in Figure 3. The symplectomorphism ω is defined on a round neighbourhood of $R(a)$. $\Box$

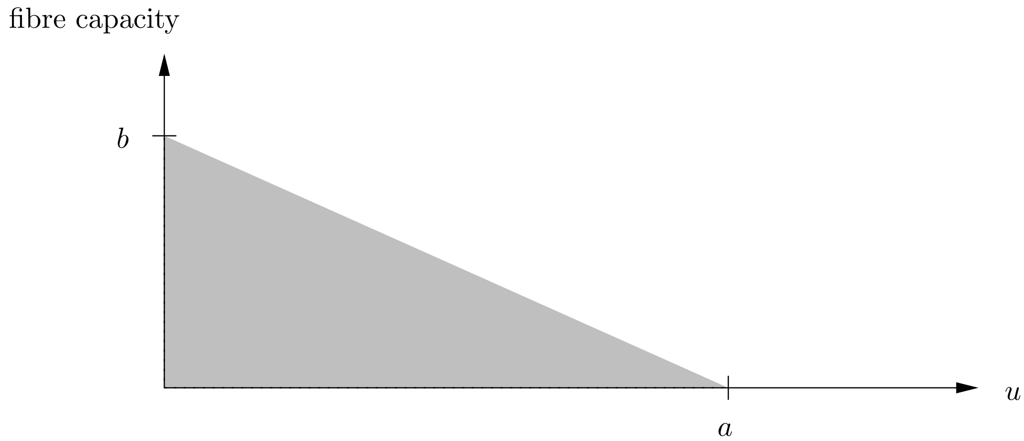

Figure 2: The trapezoid $T(a,b)$

Lemma 3.3 and its proof readily imply that in order to construct for any $a > 2\pi$ and $\epsilon > 0$ an embedding Φ satisfying (7) it is enough to find for any $a > 2\pi$ and $\epsilon > 0$ an embedding $\Psi\colon T(a,\pi) \hookrightarrow T(\frac{a}{2}+\pi+\epsilon, \frac{a}{2}+\pi+\epsilon)$, $(u,v,z_2) \mapsto (u',v',z_2')$ satisfying

$$u' + \pi|z_2'|^2 < \frac{a}{2} + \epsilon + \frac{\pi u}{a} + \pi|z_2|^2 \quad \text{for all } (u,v,z_2) \in T(a,\pi). \tag{8}$$

3.1 The folding construction

The idea in the construction of an embedding Ψ satisfying (8) is to separate the small fibres from the large ones and then to fold the two parts on top of each other.

Step 1. Following [19, Lemma 2.1] we first separate the "low" regions over $R(a)$ from the "high" ones:

Let $\delta > 0$ be small. Let $\mathcal{F}$ be described by U and f as in Figure 4 and write

$$P_1 = U \cap \left\{ u \le \frac{a}{2} + \delta \right\},$$
$$P_2 = U \cap \left\{ u \ge \frac{a+\pi}{2} + 9\delta \right\}$$
$$L = U \setminus (P_1 \cup P_2).$$

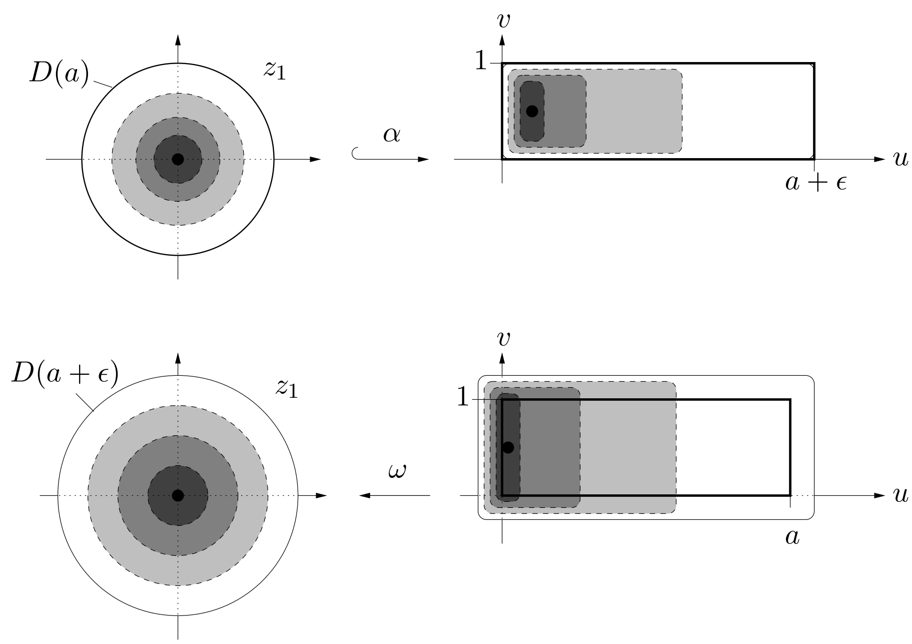

Figure 3: The first and the last base deformation

It is clear from the discussion at the beginning of the proof that there is an embedding $\beta \times id \colon T(a,\pi) \hookrightarrow \mathcal{F}$ with

$$\beta\,|_{\left\{u<\frac{a}{2}-\delta\right\}} = id \quad \text{and} \quad \beta\,|_{\left\{u>\frac{a}{2}+\delta\right\}} = id + \left(\frac{\pi}{2}+10\delta, 0\right). \tag{9}$$

Step 2. We next map the fibers into a convenient shape:

Let σ be a symplectomorphism mapping $D(\pi)$ to R_e and $D(\frac{\pi}{2})$ to R_i as specified in Figure 5. We require that for $z_2 \in D(\frac{\pi}{2})$

$$\pi|z_2|^2 + 2\delta > y(\sigma(z_2)) - \left(-\frac{\pi}{2} - 2\delta\right),$$

i.e.

$$y(\sigma(z_2)) < \pi|z_2|^2 - \frac{\pi}{2} \quad \text{for } z_2 \in D\left(\frac{\pi}{2}\right). \tag{10}$$

Write for this bundle of round squares

$$(id \times \sigma)\mathcal{F} = \mathcal{S} = \mathcal{S}(P_1) \coprod \mathcal{S}(L) \coprod \mathcal{S}(P_2).$$

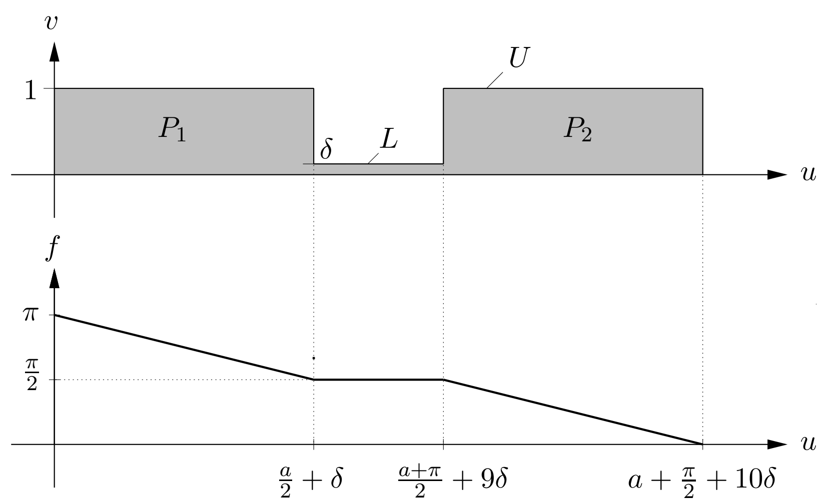

Figure 4: Separating the low fibres from the large fibres

In order to fold $\mathcal{S}(P_2)$ over $\mathcal{S}(P_1)$ we first move $\mathcal{S}(P_2)$ along the y-axis and then turn it in the z_1-direction over $\mathcal{S}(P_1)$.

Step 3. To move $\mathcal{S}(P_2)$ along the y-axis we follow again [18, p. 355]:

Let $c \in C^\infty(\mathbb{R}, \mathbb{R})$ with $c(\mathbb{R}) = [0,\, 1-\delta]$ and

$$c(t) = \begin{cases} 0, & t \le \frac{a}{2} + 2\delta \text{ and } t \ge \frac{a+\pi}{2} + 8\delta \\ 1-\delta, & \frac{a}{2} + 3\delta \le t \le \frac{a+\pi}{2} + 7\delta. \end{cases}$$

Put $I(t) = \int_0^t c(s)\, ds$ and define $\varphi \in \mathbb{C}^\infty(\mathbb{R}^4, \mathbb{R}^4)$ by

$$\varphi(u, x, v, y) = \left(u, x, v + c(u)\left(x + \frac{1}{2}\right), y + I(u) \right). \tag{11}$$

We then find

$$d\varphi(u, x, v, y) = \begin{bmatrix} \mathbb{I}_2 & 0 \\ A & \mathbb{I}_2 \end{bmatrix} \quad \text{with } A = \begin{bmatrix} * & c(u) \\ c(u) & 0 \end{bmatrix},$$

whence φ is a symplectomorphism. Moreover, with $I_\infty = I(\frac{a+\pi}{2} + 8\delta)$,

$$\varphi|_{\left\{u \le \frac{a}{2} + 2\delta\right\}} = id \quad \text{and} \quad \varphi|_{\left\{u \ge \frac{a+\pi}{2} + 8\delta\right\}} = id + (0, 0, 0, I_\infty), \tag{12}$$

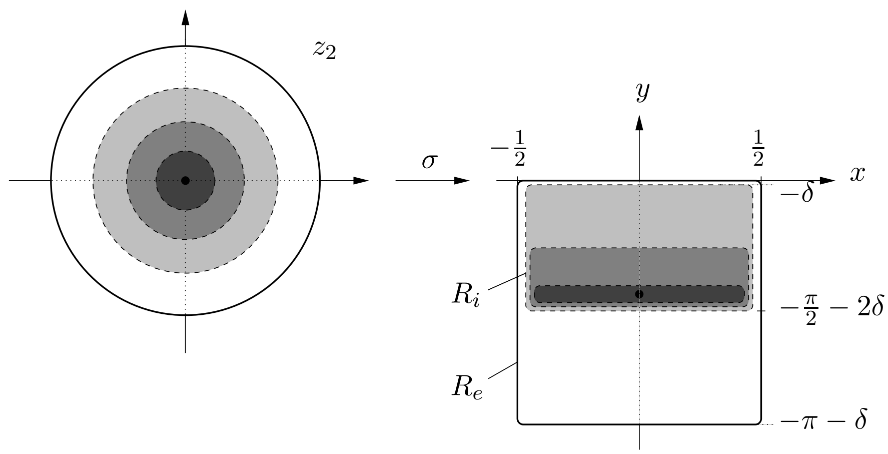

Figure 5: Preparing the fibres

and assuming that $\delta < \frac{1}{10}$ we compute

$$\frac{\pi}{2} + 2\delta < I_\infty < \frac{\pi}{2} + 5\delta. \tag{13}$$

The first inequality in (13) implies

$$\varphi(P_2 \times R_i) \cap (\mathbb{R}^2 \times R_e) = \emptyset. \tag{14}$$

Remark. φ is the crucial map of the construction; in fact, it is the only truly symplectic, i.e. not 2-dimensional map. φ is just the map which sends the lines $\{v, x, y \text{ constant}\}$ to the characteristics of the hypersurface

$$(u, x, y) \mapsto \left(u, x, c(u)\left(x + \frac{1}{2}\right), y\right),$$

which generates (the cut off of) the obvious flow separating R_i from R_e. $\diamond$

Step 4. From (11), Figure 4 and Figure 5 we read off that the projection of $\varphi(\mathcal{S})$ onto the (u, v)-plane is contained in the union of U with the open set bounded by the graph of $u \mapsto \delta + c(u)$ and the u-axis. Observe that $\delta + c(u) \leq 1$.

Define a local embedding γ of this union into the convex hull of U as follows: On P_1 the map is the identity and on P_2 it is the orientation preserving isometry between P_2 and P_1 which maps the right edge of P_2 to the

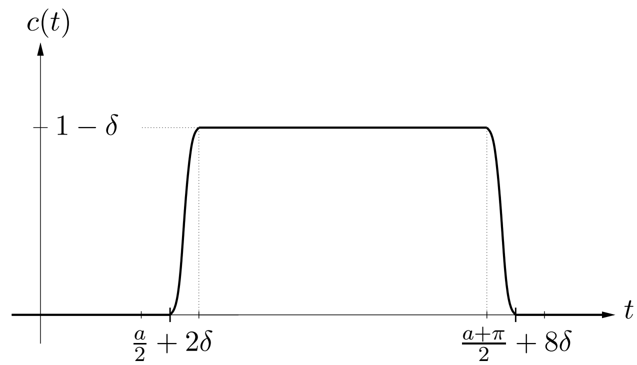

Figure 6: The cut off c

left edge of P_1. In particular, we have for $z_1 = (u, v) \in P_2$

$$u(\gamma(z_1)) = a + \frac{\pi}{2} + 10\delta - u. \tag{15}$$

On the remaining domain γ looks as follows: In a $\frac{\delta}{4}$-collar of the line from a to b the map is the identity and on a $\frac{\delta}{4}$-collar of the line from c to d the linear extension of the map on P_2, and we require

$$u'(\gamma(u,v)) - \left(\frac{a}{2} + \delta\right) < \frac{\pi}{2} + 8\delta - \left(u - \left(\frac{a}{2} + \delta\right)\right) + 2\delta,$$

i.e.

$$u'(\gamma(u,v)) < -u + \frac{\pi}{2} + a + 12\delta. \tag{16}$$

(14) shows that $\gamma \times id$ is one-to-one on $\varphi(\mathcal{S})$.

Step 5. We finally adjust the fibers:

First of all observe that the projection of $\varphi(\mathcal{S})$ onto the z_2-plane is contained in a tower shaped domain $\mathcal{T}$ (cf. Figure 8) and that by the second inequality in (13) we have $\mathcal{T} \subset \{y < \frac{\pi}{2} + 4\delta\}$.

We define a symplectomorphism τ from a neighbourhood of $\mathcal{T}$ to a disc by prescribing the preimages of concentric circles as in Figure 8: We require

- $\pi|\tau(z_2)|^2 < y + \dfrac{\pi}{2} + 3\delta \quad$ for $y \geq -\dfrac{\pi}{2} - 2\delta$ (17)
- $\pi|\tau(z_2)|^2 < \pi|\sigma^{-1}(z_2)|^2 + \dfrac{\pi}{2} + 8\delta \quad$ for $z_2 \in R_e$. (18)

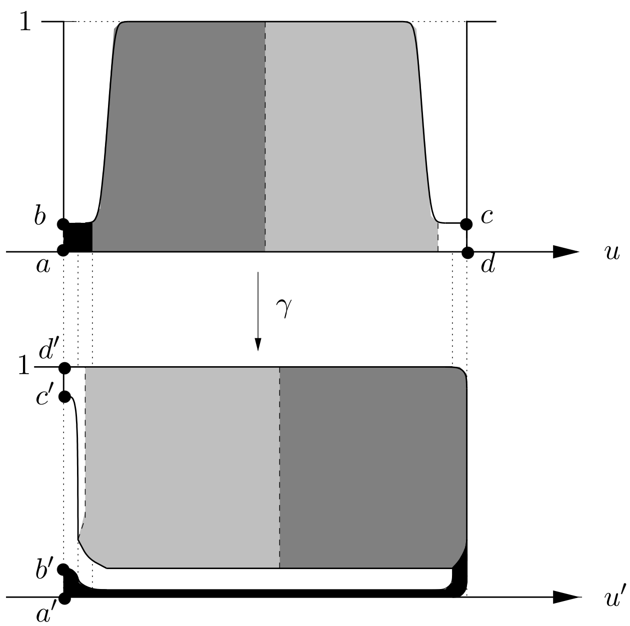

Figure 7: Folding

This finishes the construction. We think of the result as depicted in Figure 9.

Let now $\epsilon > 0$ arbitrary and choose $\delta = \min\{\frac{1}{10}, \frac{\epsilon}{14}\}$. It remains to check that

$$\Psi \stackrel{\text{def}}{=} (\gamma \times \tau) \circ \varphi \circ (\beta \times \sigma)$$

satisfies (8). So let $z = (z_1, z_2) = (u, v, x, y) \in T(a, \pi)$ and write $\Psi(z) = (u', v', z_2')$. We have to show that

$$u' - \frac{\pi u}{a} + \pi |z_2'|^2 - \pi |z_2|^2 < \frac{a}{2} + 14\delta. \tag{19}$$

Case 1. $\beta(z_1) \in P_1$:

(9) implies $u < \frac{a}{2} + \delta$, and by (12) and step 4 we have $\varphi = id$ and $\gamma = id$, whence (9) and (18) give

$$u' = u'(\beta(u, v)) < u + 2\delta,$$
$$\pi |z_2'|^2 = \pi |\tau(\sigma(z_2))|^2 < \pi |z_2|^2 + \frac{\pi}{2} + 8\delta.$$

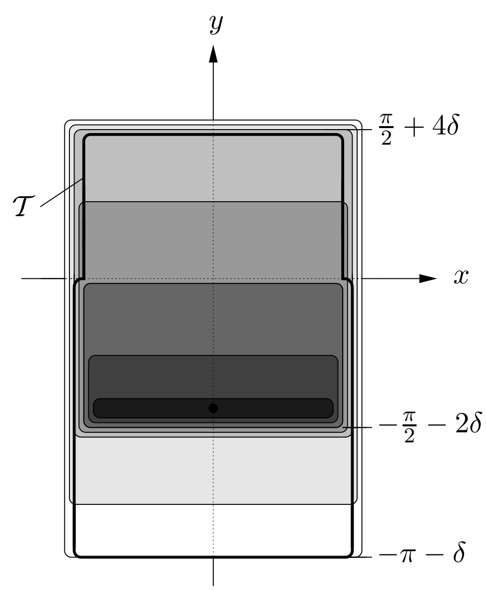

Figure 8: Mapping the tower to a disc

Therefore

$$\begin{aligned} u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 &< u\left(1 - \frac{\pi}{a}\right) + 2\delta + \frac{\pi}{2} + 8\delta \\ &< \frac{a}{2}\left(1 - \frac{\pi}{a}\right) + \frac{\pi}{2} + 11\delta \\ &= \frac{a}{2} + 11\delta. \end{aligned}$$

Case 2. $\beta(z_1) \in P_2$:

Step 2 shows $\sigma(z_2) \in R_i$, by (12) we have $\varphi = id + (0,0,0,I_\infty)$, and (9) implies $u > \frac{a}{2} - \delta$ and $u(\beta(z_1)) + 2\delta \geq u + \frac{\pi}{2} + 10\delta$, whence by (15)

$$u' = u'(\gamma(\beta(z_1))) = a + \frac{\pi}{2} + 10\delta - u(\beta(z_1)) \leq a - u + 2\delta.$$

Moreover, from (17), (10) and (13) we see

$$\begin{aligned} \pi|z_2'|^2 &= \pi|\tau(\sigma(z_2) + (0, I_\infty))|^2 \\ &< y(\sigma(z_2)) + I_\infty + \frac{\pi}{2} + 3\delta \\ &< \pi|z_2|^2 - \frac{\pi}{2} + \frac{\pi}{2} + 5\delta + \frac{\pi}{2} + 3\delta \\ &< \pi|z_2|^2 + \frac{\pi}{2} + 8\delta. \end{aligned}$$

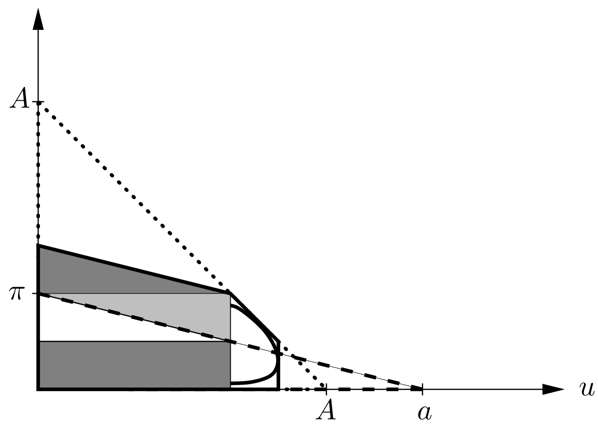

Figure 9: Folding an ellipsoid into a ball

Therefore

$$\begin{aligned} u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 &< a - u\left(1 + \frac{\pi}{a}\right) + 2\delta + \frac{\pi}{2} + 8\delta \\ &< a - \frac{a}{2}\left(1 + \frac{\pi}{a}\right) + \frac{\pi}{2} + 12\delta \\ &= \frac{a}{2} + 12\delta. \end{aligned}$$

Case 3. $\beta(z_1) \in L$:

By construction we have $\sigma(z_2) \in R_i$, and using the definition of φ, inequality (16) implies

$$u' < -u(\beta(u,v)) + \frac{\pi}{2} + a + 12\delta.$$

Next (17), (10) and the estimate $I(t) < (1-\delta)(t - (\frac{a}{2} + 2\delta))$ give

$$\begin{aligned} \pi|z_2'|^2 &< \pi|\tau(x(\sigma(z_2)), y(\sigma(z_2)) + I(u(\beta(u,v))))|^2 \\ &< y(\sigma(z_2)) + I(u(\beta(u,v)) + \frac{\pi}{2} + 3\delta \\ &< \pi|z_2|^2 - \frac{\pi}{2} + (1-\delta)\left(u(\beta(u,v)) - \frac{a}{2} - 2\delta\right) + \frac{\pi}{2} + 3\delta. \end{aligned}$$

Moreover, (9) shows $\frac{a}{2} - \delta < u < \frac{a}{2} + \delta$, whence $u(\beta(u,v)) > u > \frac{a}{2} - \delta$, and

therefore

$$\begin{aligned}
u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 &< -u(\beta(u,v)) + \frac{\pi}{2} + a + 12\delta - \frac{\pi}{a}\left(\frac{a}{2} - \delta\right) \\
&\quad +u(\beta(u,v)) - \frac{a}{2} - 2\delta - \delta\left(\frac{a}{2} - \delta\right) + \frac{a}{2}\delta + 2\delta^2 + 3\delta \\
&= \frac{a}{2} + 13\delta + \frac{\pi}{a}\delta + 3\delta^2 \\
&< \frac{a}{2} + 14\delta.
\end{aligned}$$

□

3.2 Folding in four dimensions

In four dimensions we may exploit the great flexibility of symplectic maps which only depend on the fibre coordinates to provide rather satisfactory embedding results for simple shapes.

We first discuss a modification of the folding construction described in the previous section, then explain multiple folding and finally calculate the optimal embeddings of ellipsoids and polydiscs into balls and cubes which can be obtained by these methods.

Not to disturb the exposition furthermore with δ-terms we skip them in the sequel. Since all sets considered will be bounded and all constructions will involve only finitely many steps, we won't lose control of them.

3.2.1 The folding construction in four dimensions

The map σ in step 2 of the folding construction given in the previous section was dictated by the estimate (19) necessary for the n-dimensional result. As a consequence, the map γ had to disjoin the z_2-projection of P_2 from the one of P_1, and we ended up with the unused white sandwiched triangle in Figure 9. In order to use this room as well we modify the construction as follows:

Replace the map σ of step 2 by the map σ given by Figure 10. If we define φ as in (11), the z_2-projection of the image of φ will almost coincide with the image of σ. Choose now γ as in step 4 and define the final map τ on a neighbourhood of the image of φ such that it restricts to σ^{-1} on the image of σ. If all the δ's were chosen appropriately, the composite map Ψ will be one-to-one, and the image Ψ will be contained in $T(a/2+\pi+\epsilon,\, a/2+\pi+\epsilon)$ for some small ϵ. We think of the result as depicted in Figure 11.

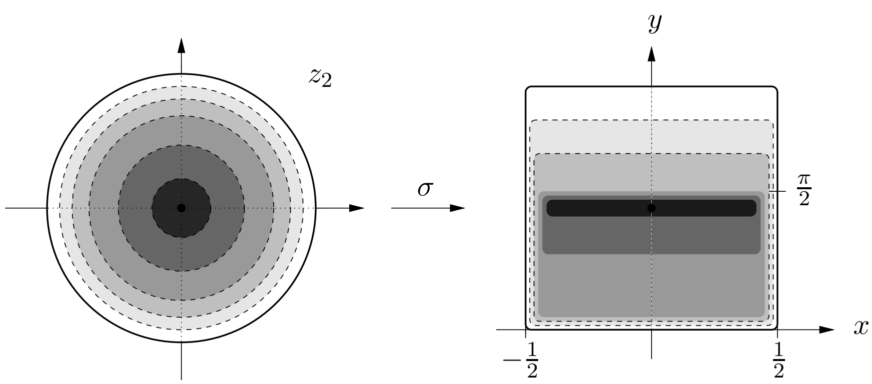

Figure 10: The modified map σ

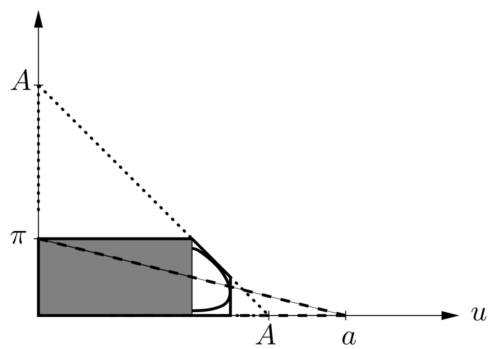

Figure 11: Folding in four dimensions

3.2.2 Multiple folding

Neither Theorem 2 nor Traynor's theorem stated at the beginning of section 3 tells us if $E(\pi, 4\pi)$ embeds in $B^4(a)$ for some $a \leq 3\pi$ (cf. Figure 1). Multiple folding, which is explained in this subsection, will provide better embeddings. To understand the general construction it is enough to look at a 3-fold: The folding map Ψ is the composition of maps explained in Figure 12. Here are the details: Pick reasonable $u_1, \dots, u_4 \in \mathbb{R}_{>0}$ with $\sum_{j=1}^4 u_i = a$ and put

$$l_i = \pi - \frac{\pi}{a} \sum_{j=1}^{i} u_j, \qquad i = 1, 2, 3. \tag{20}$$

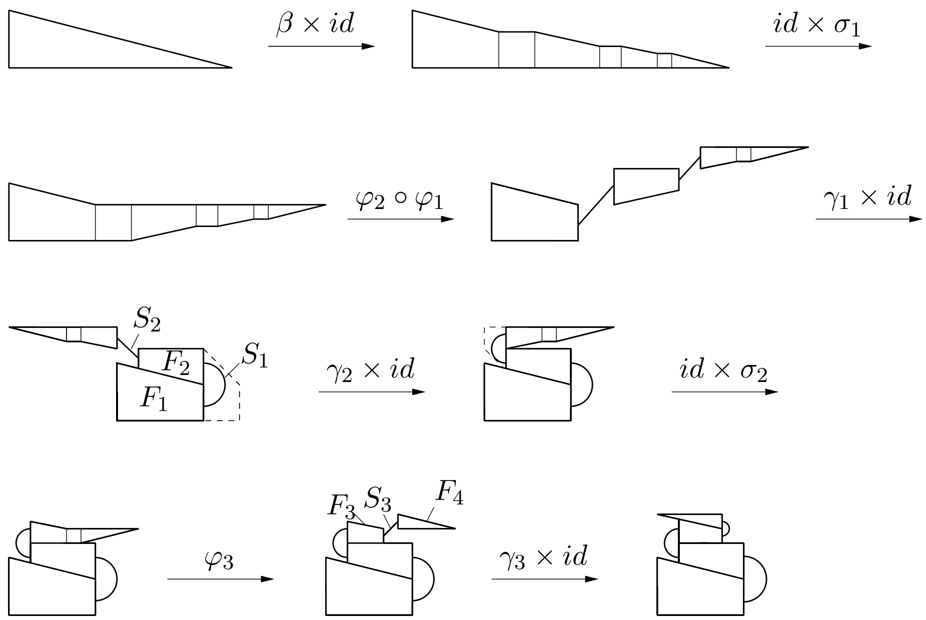

Figure 12: Multiple folding

Step 1. Define $\beta\colon R(a) \to U$ by Figure 13.

Step 2. For $l_1 = \pi/2$ the map σ_1 is given by Figure 10, and in general it is defined to be the symplectomorphism from $D(\pi)$ to the left round rectangle in Figure 14.

Step 3. Choose cut offs c_i over L_i, $i = 1, 2$, put $I_i(t) = \int_0^t c_i(s)\,ds$ and define φ_i on $\beta \times \sigma_1(T(a, \pi))$ by

$$\varphi_i(u, x, v, y) = \left(u, x, v + c_i(u)\left(x + \frac{1}{2}\right), y + I_i(u)\right).$$

The effect of $\varphi_2 \circ \varphi_1$ on the fibres is explained by Figure 14.

Step 4. γ_1 is essentially the map γ of the folding construction: On P_1 it is the identity, for $u_1 \le u \le u_1 + l_1$ it looks like the map in Figure 7, and for $u > u_1 + l_1$ it is an isometry. Observe that by construction, the slope of the stairs S_2 is 1, while the one of the upper edge of the floor F_1 is less than 1. S_2 and F_1 are thus disjoint.

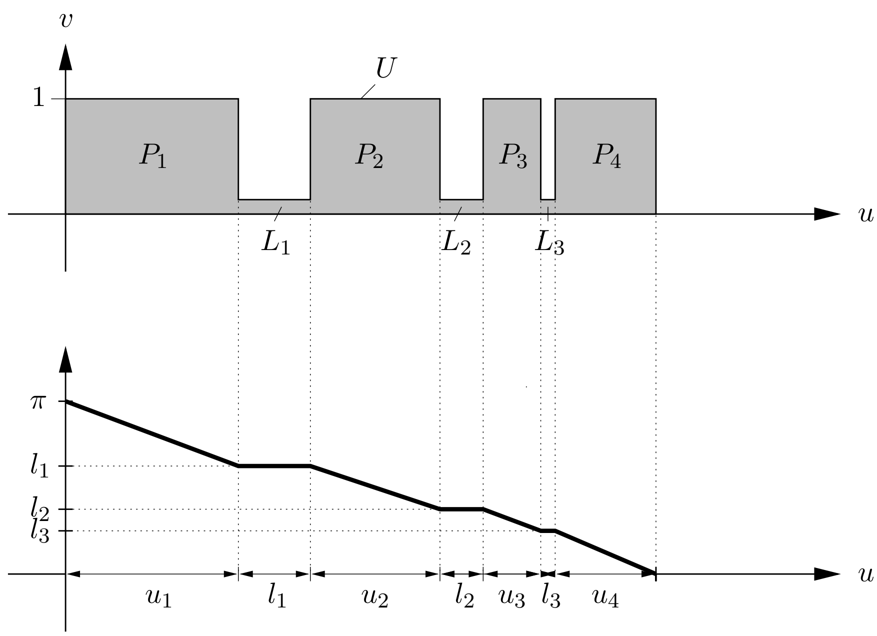

Figure 13: β

Step 5. $\gamma_2 \times id$ is not really a global product map, but restricts to a product on certain pieces of its domain: It fixes $F_1 \coprod S_1 \coprod F_2$, and it is the product $\gamma_2 \times id$ on the remaining domain where γ_2 restricts to an isometry on $u_1 \leq 0$ and looks like the map given by Figure 15 on the z_1-projection of S_2.

For further reference, we summarize the result of the two preceding steps in the

Folding Lemma. *Let S be the stairs connecting two floors of minimal respectively maximal height l.*

(i) If the floors have been folded on top of each other by folding on the right, S is contained in a trapezoid with horizontal lower edge of length l and left respectively right edge of length $2l$ respectively l.

(ii) If the floors have been folded on top of each other by folding on the left, S is contained in a trapezoid with horizontal upper edge of length l and left respectively right edge of length l respectively $2l$.

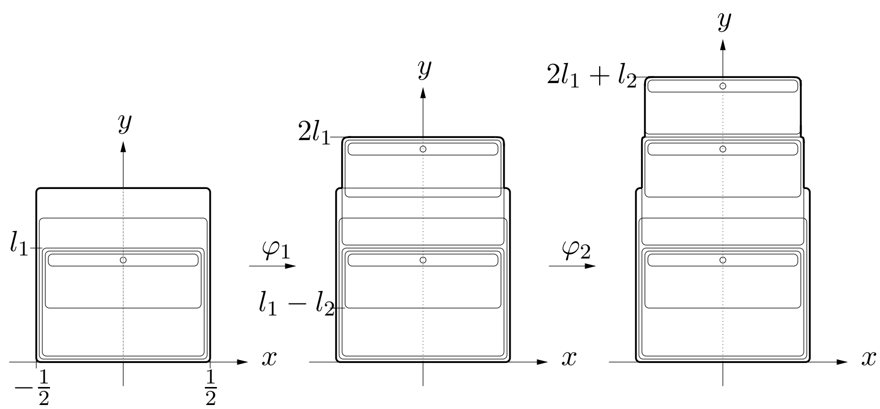

Figure 14: The first and the second lift

The remaining three maps are restrictions to the relevant parts of already considered maps.

Step 6. On $\{y > 2l_1\}$ the map σ_2 is the automorphism whose image is described by the same scheme as the image of σ_1, and $id \times \sigma_2$ restricts to the identity everywhere else.

Step 7. On $\{y > 2l_1\}$ the map φ_3 restricts to the usual lift, and it is the identity everywhere else.

Step 8. Finally, $\gamma_3 \times id$ turns F_4 over F_3. It is an isometry on F_4, looks like the map given by Figure 7 on S_3 and restricts to the identity everywhere else.

This finishes the multiple folding construction.

3.2.3 Embeddings into balls

In this subsection we use multiple folding to construct good embeddings of ellipsoids into balls, and we also look at embeddings of polydiscs into balls.

3.2.3.1 Embedding ellipsoids into balls We now choose the u_j's optimal.

Fix $u_1 > 0$. As proposed in Figure 31, we assume that the second floor F_2 touches the boundary of $T(A, A)$ and that all the other u_j's are chosen

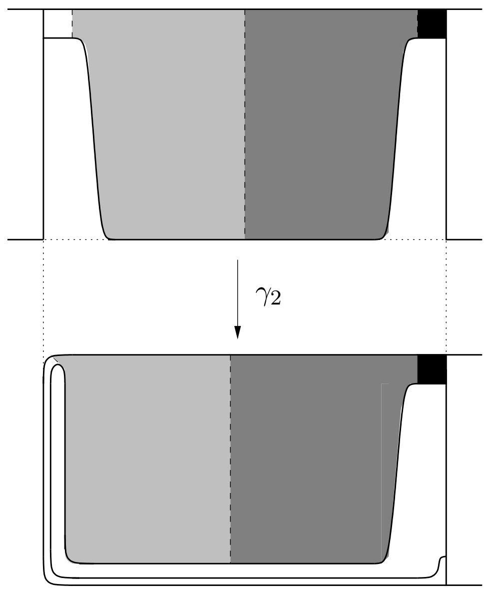

Figure 15: Folding on the left

maximal. In other words, A is given by

$$A(a, u_1) = u_1 + 2\, l_1 = 2\pi + \left(1 - \frac{2\pi}{a}\right) u_1, \tag{21}$$

and we proceed as follows: If the remaining length $r_1 = a - u_1$ is smaller than u_1, i.e. $u_1 \geq a/2$, we are done; otherwise we try to fold a second time. By the Folding Lemma, this is possible if and only if $l_1 < u_1$, i.e.

$$u_1 > \frac{a\pi}{a + \pi}. \tag{22}$$

If (22) does not hold, the embedding attempt fails; if (22) holds, the Folding Lemma and the maximality of u_2 imply $u_2 = u_1 - l_2$, whence by (20)

$$u_2 = \frac{a+\pi}{a-\pi} u_1 - \frac{a\pi}{a-\pi}.$$

If the upper left corner of F_3 lies outside $T(A, A)$, the embedding attempt fails, otherwise we go on.

In general, assume that we folded already j times and that j is even. If the length of the remainder $r_j = r_{j-1} - u_j$ is smaller than u_j, we are done; if

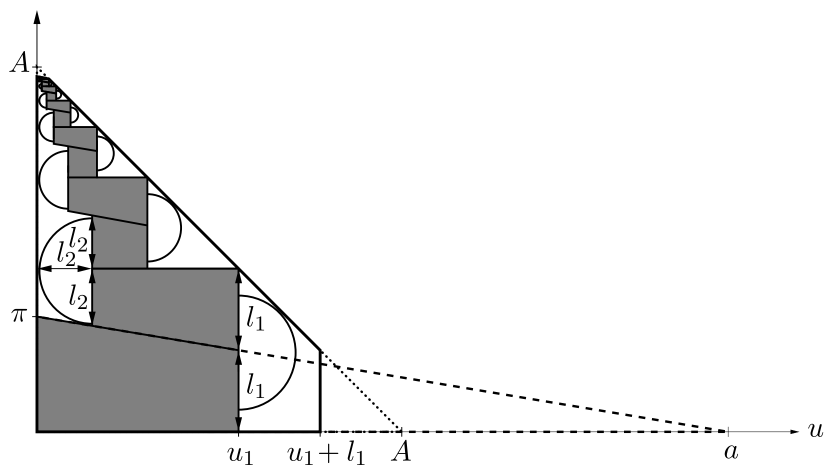

Figure 16: A 12-fold

not, we try to fold again: The Folding Lemma and the maximality of u_{j+1} imply $u_{j+1} + 2l_{j+1} = u_j$, and substituting $l_{j+1} = l_j - u_{j+1}\pi/a$ we get

$$u_{j+1} = \frac{a}{a-2\pi}(u_j - 2l_j).$$

If $u_j \leq 2l_j$, the embedding attempt fails, otherwise we go on: If the length of the new remainder $r_{j+1} = r_j - u_{j+1}$ is smaller than $u_{j+1} + l_j$, we are done; otherwise we try to fold again: The Folding Lemma and the maximality of u_{j+2} imply $u_{j+2} + l_{j+2} = u_{j+1} + l_j$, whence by (20)

$$u_{j+2} = \frac{a+\pi}{a-\pi} u_{j+1}.$$

The embedding attempt fails here if and only if the upper left corner of the floor F_{j+3} lies outside $T(A, A)$; if this does not happen, we may go on as before.

First of all note that whenever the above embedding attempt succeeds, it indeed describes an embedding of $E(\pi, a)$ into $T(A(a, u_1), A(a, u_1))$. In fact, it is enough to define the fiber adjusting map τ on a small neighbourhood of the resulting tower $\mathcal{T}$ in such a way that for any $z_2 = (x, y)$, $z_2' = (x', y') \in \mathcal{T}$ we have

$$y \leq y' \implies |\tau(z_2)|^2 < |\tau(z_2')|^2.$$

(21) shows that we have to look for the smallest u_1 for which the above embedding attempt succeeds. Call it $u_0 = u_0(a)$. As we have seen above, u_0 lies in the interval

$$I_a = \left[\frac{a\pi}{a+\pi}, \frac{a}{2}\right]. \tag{23}$$

Moreover, if the embedding attempt succeeds for u_1, the same clearly holds true for any $u_1' > u_1$. Hence, given $u_1 \in I_a$, the corresponding embedding attempt succeeds if and only if $u_1 \geq u_0$. Appendix A1 provides a computer program calculating u_0, and the result $s_{EB}(a) = 2\pi + (1 - 2\pi/a)u_0$ is discussed and compared with the one yielded by Lagrangian folding in subsection 3.5.

Remarks. 1. Simple geometric considerations show that our choices in the above algorithm are optimal, i.e. $s_{EB}(a)$ provides the best estimate for an embedding of $E(\pi, a)$ into a ball obtainable by multiple folding.

2. Let $u_1 > u_0$ and let $N(u_1)$ be the number of folds needed in the above embedding procedure determined by u_1. Then $N(u_1) \to \infty$ as $u_1 \searrow u_0$, i.e. the best embeddings are obtained by folding arbitrarily many times. This follows again from an easy geometric reasoning.

3. Fix N and let $A_N(a)$ be the function describing the optimal embedding obtainable by folding N times. Then $\{A_N\}_{n\in\mathbb{N}}$ is a monoton decreasing family of rational functions on $[2\pi, \infty[$. For instance,

$$A_1(a) = 2\pi + (a - 2\pi)\frac{1}{2}, \qquad A_2(a) = 2\pi + (a - 2\pi)\frac{a+\pi}{3a+\pi}$$

and

$$A_3(a) = 2\pi + (a - 2\pi)\frac{(a+\pi)(a+2\pi)}{4(a^2 + a\pi + \pi^2)}.$$

So, $A_1'(2\pi) = \frac{1}{2}$ and $A_2'(2\pi) = A_3'(2\pi) = \frac{3}{7}$. One can show that $A_N'(2\pi) = \frac{3}{7}$ for all $N \geq 3$. Thus

$$\limsup_{\epsilon \to 0^+} \frac{s_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} = \frac{3}{7}.$$

$\diamond$

3.2.3.2 Embedding polydiscs into balls

Proposition 3.4 *Let $a > 2\pi$ and $\epsilon > 0$. Then $P(\pi, a)$ embeds in $B^4(s_{PB}(a) + \epsilon)$, where s_{PB} is given by*

$$s_{PB}(a) = \frac{a - 2\pi}{2k} + (k+2)\pi, \qquad 2(k^2 - k + 1) < a/\pi \leq 2(k^2 + k + 1).$$

Proof. Let $N = 2k - 1$, $k \in \mathbb{N}$, be odd. From Figure 17 we read off that

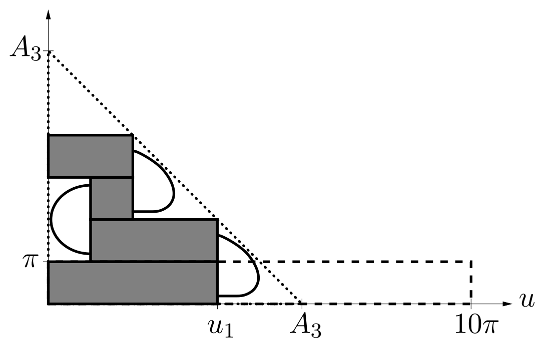

Figure 17: The optimal embedding $P(\pi, 10\pi) \hookrightarrow B^4(A)$

under the condition $u_1 > N\pi$ the optimal embedding by folding N times is described by

$$\begin{aligned} a &= \pi + 2(u_1 - \pi) + 2(u_1 - 3\pi) + \cdots + 2(u_1 - N\pi) + \pi \\ &= 2\pi + 2ku_1 - 2k^2\pi \end{aligned}$$

and $A_N(a) = u_1 + 2\pi$; hence

$$A_N(a) = \frac{a - 2\pi}{2k} + (k+2)\pi,$$

provided that $A_N(a) - 2\pi > (2k-1)\pi$. This condition translates to $a > 2(k^2 - k + 1)\pi$, and the claim follows. $\square$

Remark. s_{PB} is the optimal result obtainable by multiple folding. In fact, a simple geometric argument or a similar calculation as in the proof shows that folding $2k$ times yields worse estimates. $\diamond$

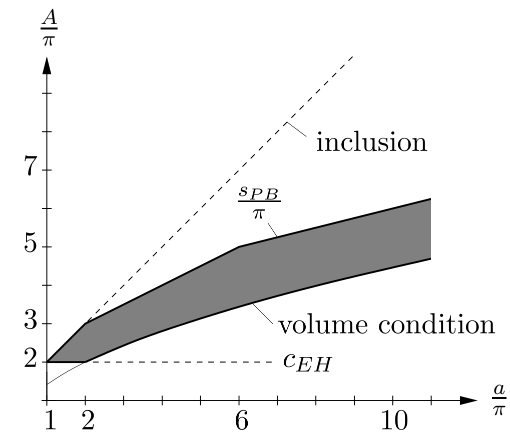

Figure 18: What is known about $P(\pi, a) \hookrightarrow B^4(A)$

Remark 3.5 Let $d_{PB}(a) = s_{PB}(a) - \sqrt{2\pi a}$ be the difference between s_{PB} and the volume condition. d_{PB} attains its local maxima at $a_k = 2(k^2 - k+1)\pi$, where $d_{PB}(a_k) = (2k+1)\pi - 2\pi\sqrt{k^2-k+1}$. This is an increasing sequence converging to 2π. $\diamond$

3.2.4 Embeddings into cubes

Given an open set U in $\mathbb{C}^n$, call the orthogonal projections of U onto the n symplectic coordinate planes the shadows of U. As pointed out in [10, p. 580], symplectic capacities measure to some extent the areas of the shadows of a set. Of course, this can not be made rigorous since the areas of shadows are no symplectic invariants, but for sufficiently regular sets these areas indeed are symplectic capacities: As remarked before, the capacities $a_1, \ldots, a_n$ of the ellipsoid $E(a_1, \ldots, a_n)$ are symplectic capacities and, more generally, given any bounded U with connected smooth boundary ∂U of restricted contact type and with a shadow whose boundary is the shadow of a closed characteristic on ∂U which lies in a single symplectic coordinate direction, this shadow is a capacity of U [7, Proposition 2]. Moreover, the smallest shadow of a polydisc and of a symplectic cylinder are capacities.

Instead of studying embeddings into minimal balls, i.e. to reduce the diameter of a set, it is therefore a more symplectic enterprise to look for minimal embeddings into a polydisc $C^{2n}(a)$, i.e. to reduce the maximal shadow.

The Non-Squeezing Theorem states that the smallest shadow of simple sets (like ellipsoids, polydiscs or cylinders) can not be reduced. We therefore call obstructions to the reduction of the maximal shadow highest order rigidity. (More generally, calling an ellipsoid or a polydisc given by $a_1 \leq \cdots \leq a_n$ i-reducible if there is an embedding into $C^{2i}(a') \times \mathbb{R}^{2n-2i}$ for some $a' < a_i$, one might explore i-th order rigidity.)

The disadvantage of this approach to higher order rigidity is that for a polydisc there are no good higher invariants available, in fact, Ekeland-Hofer-capacities see only the smallest shadow [7, Proposition 5]:

$$c_j(P(a_1, \dots, a_n)) = j a_1.$$

Many of the polydisc-analogues of the rigidity results for ellipsoids proved in section 2 are therefore either wrong or much harder to prove. It is for instance not true that $P(a_1, \dots, a_n)$ embeds linearly in $P(a_1', \dots, a_n')$ if and only if $a_i \leq a_i'$ for all i, for a long enough 4-polydisc may be turned into the diagonal of a cube of smaller maximal shadow:

Lemma 3.6 *Let $r > 1 + \sqrt{2}$. Then $P^{2n}(\pi, \dots, \pi, \pi r^2)$ embeds linearly in $C^{2n}(a)$ for some $a < \pi r^2$.*

Proof. It is clearly enough to prove the lemma for $n = 2$. Consider the linear symplectomorphism given by

$$(z_1, z_2) \mapsto (z_1', z_2') = \frac{1}{\sqrt{2}}(z_1 + z_2, z_1 - z_2).$$

For $(z_1, z_2) \in P(\pi, \pi r^2)$ we have for $i = 1, 2$

$$|z_i'|^2 \leq \frac{1}{2}(|z_1|^2 + |z_2|^2 + 2|z_1||z_2|) \leq \frac{1}{2} + \frac{r^2}{2} + r, \tag{24}$$

and the right hand side of (24) is strictly smaller than r^2 provided that $r > 1 + \sqrt{2}$. □

Similarly, we don't know how to prove the full analogue of Proposition 2.1:

Let $\mathcal{P}(n)$ be the collection of polydiscs

$$\mathcal{P}(n) = \{P(a_1, \dots, a_n)\}$$

and write $\preceq_i$ for the restrictions of the relations $\leq_i$ to $\mathcal{P}(n)$. Again $\preceq_2$ and $\preceq_3$ are very similar, again all the relations $\preceq_i$ are clearly reflexive and

transitive, and again the identitivity of $\preceq_2$, which again implies the one of $\preceq_1$, follows from the equality of the spectra, which is implied by the equality of the volumes. (Observe that, even though the boundary of a polydisc is not smooth, its spectrum is still well defined.) For n=2 the identitivity of $\preceq_3$ is seen by using any symplectic capacity, which determines the smallest shadow, and the equality of the volumes; but for arbitrary n we don't know a proof.

While the lack of convenient invariants made it impossible to get good rigidity results for embeddings into polydiscs, the folding construction provides us with rather satisfactory flexibility results.

3.2.4.1 Embedding ellipsoids into cubes We again use the notation of section 3.2, fold first at some reasonable u_1 and then choose the subsequent u_j's maximal (see Figure 19). Let $w(a, u_1) = u_1 + l_1 = \pi + (1 - \pi/a)u_1$ be

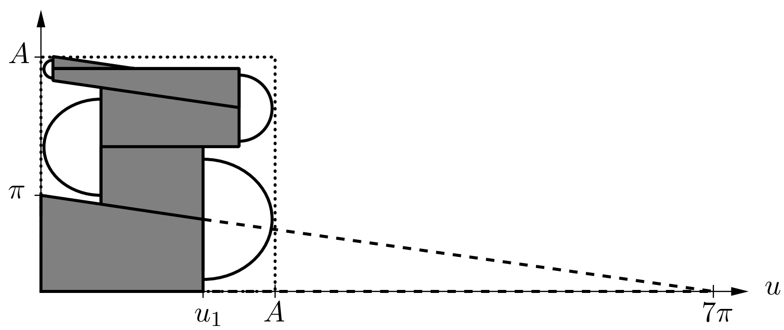

Figure 19: The optimal embedding $E(\pi, 7\pi) \hookrightarrow C^4(A)$

the width of the image and $h = h(a, u_1)$ its height.

Let's first see what we get by folding once: The only condition on u_1 is $a/2 \leq u_1$, whence $h(a, u_1) = \pi < \pi + (1 - \pi/a)u_1 = w(a, u_1)$. The optimal choice of u_1 is thus $u_1 = a/2$.

Suppose now that we fold at least twice. The only condition on u_1 is then again $l_1 < u_1$, i.e.

$$u_1 > \frac{a\pi}{a + \pi}.$$

Observe that $h(a, u_1)$ diverges if u_1 approaches $a\pi/(a+\pi)$. Note also that w is increasing in u_1 while h is decreasing. Thus, $w(a, u_1)$ and $h(a, u_1)$ intersect exactly once, namely in the optimal u_1, which we call u_0. In particular,

we see that folding only once never yields an optimal embedding. Write $s_{EC}(a) = \pi + (1 - \pi/a)u_0$ for the resulting estimate. It is computed in Appendix A2. Again, it is easy to see that our choices in the above procedure are optimal, i.e. $s_{EC}(a)$ provides the best estimate for an embedding of $E(\pi, a)$ into a cube obtainable by symplectic folding.

Example. If we fold exactly twice, we have $h = 2l_1 + l_2$, or, since l_2 satisfies $a = u_1 + u_2 + (a/\pi)l_2$ and $u_2 = u_1 - l_2$,

$$h = 2\pi - \frac{2\pi}{a}u_1 + \frac{\pi(a - 2u_1)}{a - \pi}.$$

Thus, provided that $l_2 + (a/\pi)l_2 \leq w$, the equation $h = w$ yields

$$u_0 = \frac{a\pi(2a - \pi)}{a^2 + 2a\pi - \pi^2}. \tag{25}$$

Indeed, u_0 satisfies (25) whenever $a > \pi$. Finally, $l_2 + (a/\pi)l_2 \leq w$ holds if and only if $\pi \leq a \leq 3\pi$. $\diamond$

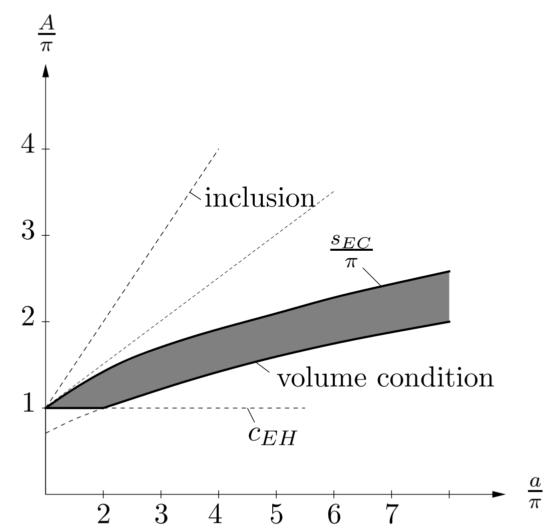

Figure 20: What is known about $E(\pi, a) \hookrightarrow C^4(A)$

In fact, (25) also holds true for all a for which the optimal embedding of $E(\pi, a)$ obtainable by multiple folding is a 3-fold for which the height is still described by $h = 2l_1 + l_2$, i.e. for which $u_4 \leq u_3$. This happens for

$3 < a/\pi < 4.2360\ldots$, whence

$$s_{EC}(a) = \frac{a\pi(3a-\pi)}{a^2+2a\pi-\pi^2} \quad \text{for } 1 \le \frac{a}{\pi} \le 4.2360\ldots.$$

In general, s_{EC} is a piecewise rational function. Its singularities are those a for which $u_{N(a)} = u_{N(a)+1}$, where we wrote $N(a)$ for the number of folds determined by $u_0(a)$.

Remark 3.7 Let $d_{EC}(a) = s_{EC}(a) - \sqrt{\pi a/2}$ be the difference between s_{EC} and the volume condition. The set of local minima of d_{EC} coincides with its singular set, i.e. with the singular set of s_{EC}. On the other hand, d_{EC} attains its local maxima at those a for which the point of $F_{N(a)+1}$ touches the boundary of $T(A,A)$. Computer calculations suggest that on this set, d_{EC} is increasing, but bounded by $(2/3)\pi$. $\diamond$

3.2.4.2 Embedding polydiscs into cubes

Proposition 3.8 *Let $a > 2\pi$ and $\epsilon > 0$. Then $P(\pi,a)$ embeds in $C^4(s_{PC}(a)+\epsilon)$, where s_{PC} is given by*

$$s_{PC}(a) = \begin{cases} (N+1)\pi, & (N-1)N+2 < \frac{a}{\pi} \le N^2+1 \\ \frac{a+2N\pi}{N+1}, & N^2+1 < \frac{a}{\pi} \le N(N+1)+2\,. \end{cases}$$

Proof. The optimal embedding by folding N times is described by

$$2u_1 + (N-1)(u_1-\pi) = a,$$

whence $u_1 = \frac{a+(N-1)\pi}{N+1}$; in fact, by the assumption on a, the only condition $u_1 > \pi$ for $N \ge 2$ is satisfied. Thus $A_N(a) = \max\{\frac{a+2N\pi}{N+1}, (N+1)\pi\}$, and the proposition follows. $\square$

Remark 3.9 The difference $d_{PC}(a) = s_{PC}(a) - \sqrt{\pi a}$ between s_{PC} and the volume condition attains its local maxima at $a_N = (N^2-N+2)\pi$, where $d_{PC}(a_N) = (N+1)\pi - \sqrt{N^2-N+2}\,\pi$. This is an increasing sequence converging to $(3/2)\pi$. $\diamond$

Since for $a \le 2\pi$ folding cannot reduce $P(\pi,a)$ and since we believe that for small a folding is essentially the only way to achieve a reduction (see also [20]), we state:

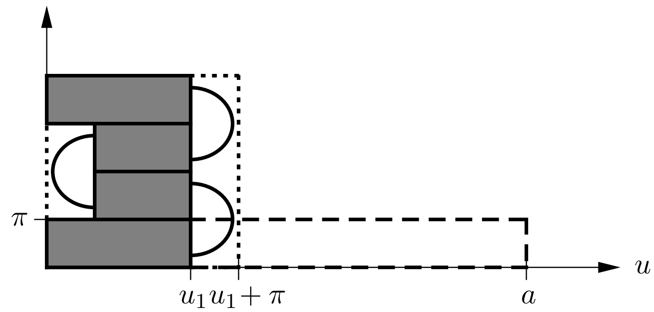

Figure 21: Folding $P(\pi, a)$ three times

Conjecture 3.10 *The polydisc-analogue of Theorem 1' holds. In particular,*
$P^{2n}(\pi, \dots, \pi, a)$ *embeds in* $C^{2n}(A)$ *for some* $A < a$ *if and only if* $a > 2\pi$.

3.3 Folding in higher dimensions

Even though symplectic folding is an essentially four dimensional process, we may still use it to get good embeddings in higher dimensions as well. The point is that we may fold into different symplectic directions of the fiber. In view of the applications of higher dimensional folding in subsection 4.1 and 4.2 we will concentrate on embedding skinny polydiscs into cubes and skinny ellipsoids into balls and cubes.

Given domains $U \subset \mathbb{R}^{2n}$ and $V, W \subset \mathbb{R}^n$ and given $\alpha > 0$, we set

$$\alpha U = \{\alpha z \in \mathbb{R}^{2n} \mid z \in U\} \quad \text{and} \quad \alpha V \times W = \alpha(V \times W).$$

As in the four dimensional case we may view an ellipsoid $E(a_1, \dots, a_n)$ as fibered over the disc $D(a_n)$ with ellipsoids $\gamma E(a_1, \dots, a_{n-1})$ of varying size as fibres. By deforming the base $D(a_n)$ to a rectangle as in Figure 3 we may get rid of the y_1-coordinate. It will be convenient to get rid of the other y_i-coordinates too. Write $\mathbb{R}^{2n}(x, y) = \mathbb{R}^n(x) \times \mathbb{R}^n(y)$ and set

$$\triangle(a_1, \dots, a_n) = \{0 < x_1, \dots, x_n \mid \sum_{i=1}^n \frac{x_i}{a_i} < 1\} \subset \mathbb{R}^n(x),$$
$$\square(b_1, \dots, b_n) = \{0 < y_i < b_i,\ 1 \le i \le n\} \subset \mathbb{R}^n(y).$$

Lemma 3.11 *For all $\epsilon > 0$,*

(i) $E(a_1-\epsilon,\dots,a_n-\epsilon)$ embeds in $\triangle(a_1,\dots,a_n)\times\square^n(1)$ in such a way that for all $\alpha\in\,]0,1]$, $\alpha E(a_1-\epsilon,\dots,a_n-\epsilon)$ is mapped into $(\alpha+\epsilon)\triangle(a_1,\dots,a_n)\times\square^n(1)$.

(ii) $\triangle(a_1-\epsilon,\dots,a_n-\epsilon)\times\square^n(1)$ embeds in $E(a_1,\dots,a_n)$ in such a way that for all $\alpha\in\,]0,1]$, $\alpha\triangle(a_1-\epsilon,\dots,a_n-\epsilon)\times\square^n(1)$ is mapped into $(\alpha+\epsilon)E(a_1,\dots,a_n)$.

Proof. By Lemma 3.2 we find embeddings $\alpha_i\colon D(a_i-\epsilon)\hookrightarrow\square(a_i,1)$ satisfying

$$x_i(\alpha_i(z_i)) < \pi|z_i|^2+\frac{\epsilon}{n}\frac{a_1}{\max(1,a_n)}\qquad\text{for } z_i\in D(a_i-\epsilon),\ 1\le i\le n$$

(cf. Figure 3). Given $(z_1,\dots,z_n)\in E(a_1-\epsilon,\dots,a_n-\epsilon)$ we then find

$$\begin{aligned}\sum_{i=1}^n\frac{x_i(\alpha_i(z_i))}{a_i} &< \sum_{i=1}^n\frac{\pi|z_i|^2}{a_i}+\frac{1}{a_i}\frac{\epsilon}{n}\frac{a_1}{a_n}\\ &< \max_i\frac{a_i-\epsilon}{a_i}+\frac{\epsilon}{a_n}\;=\;1-\frac{\epsilon}{a_n}+\frac{\epsilon}{a_n}\;=\;1,\end{aligned}$$

and given $(z_1,\dots,z_n)\in\alpha E(a_1-\epsilon,\dots,a_n-\epsilon)$ we find

$$\sum_{i=1}^n\frac{x_i(\alpha_i(z_i))}{a_i}\;<\;\sum_{i=1}^n\frac{\pi|z_i|^2}{a_i}+\frac{a_1}{a_i}\frac{\epsilon}{n}\;<\;\alpha+\epsilon.$$

The proof of (ii), which uses products of maps ω_i as in Figure 3, is similar. $\square$

Forgetting about all the ϵ's, we may thus view an ellipsoid as a Lagrangian product of a simplex and a cube. In the setting of symplectic folding, however, we will still rather think of $E(a_1,\dots,a_n)$ as fibered over the base $\square(a_n,1)$. By Lemma 3.11(i) we may assume that the fiber over (x_n,y_n) is $(1-x_1/a_n)\triangle(a_1,\dots,a_{n-1})\times\square^{n-1}(1)$.

Similarly, by mapping the discs $D(a_i)$ symplectomorphically to the rectangles $\square(a_i,1)$ and then looking at the Lagrangian instead of the symplectic splitting, we may think of $P(a_1,\dots,a_n)$ as $\square(a_1,\dots,a_n)\times\square^n(1)$.

3.3.1 Embeddings of polydiscs

We fold a polydisc $P(a_1, \dots, a_n)$ by folding a four dimensional factor $P(a_i, a_j)$ for some $i \neq j \in \{1, \dots, n\}$ and leaving the other factor alone. An already folded polydisc may be folded again by restricting the folding process to a component containing no stairs. The choice of i and j is only restricted by the condition that the new image should still be embedded.

3.3.1.1 Embedding polydiscs into cubes In view of an application in subsection 4.1 we are particularly interested in embedding thin polydiscs into cubes. So fix $P^{2n}(a, \pi, \dots, \pi)$ and let A be reasonably large. As explained above, we think of $P^{2n}(a, \pi, \dots, \pi)$ as $\square^n(a, \pi, \dots, \pi) \times \square^n(1)$ and of $C^{2n}(A)$ as $\square^n(A) \times \square^n(1)$. The base direction will thus be the z_1-direction. Folding into the z_i-direction for some $i \in \{2, \dots, n\}$, we will always lift into the x_i-direction.

We describe the process for $n = 3$: First, fill a z_1-z_2-layer as well as possible by lifting N times into the x_2-direction (cf. Figure 21). Then lift once into the x_3-direction and fill a second z_1-z_2-layer .... If u_1 is chosen appropriately, we will fold N times into the x_3-direction and fill $N+1$ z_1-z_2-layers.

The following proposition generalizes Proposition 3.8 to arbitrary dimension.

Proposition 3.12 *Let $a > 2\pi$ and $\epsilon > 0$. Then $P^{2n}(\pi, \dots, \pi, a)$ embeds in $C^{2n}(s^{2n}_{PC}(a) + \epsilon)$, where s^{2n}_{PC} is given by*

$$s^{2n}_{PC}(a) = \begin{cases} (N+1)\pi, & (N-1)N^{n-1} < \frac{a}{\pi} - 2 \leq (N-1)(N+1)^{n-1} \\ \frac{a-2\pi}{(N+1)^{n-1}} + 2\pi, & (N-1)(N+1)^{n-1} < \frac{a}{\pi} - 2 \leq N(N+1)^{n-1} \, . \end{cases}$$

Proof. The optimal embedding by folding N times in each z_1-z_2-layer is described by

$$2u_1 + ((N+1)^{n-1} - 2)(u_1 - \pi) = a,$$

whence

$$u_1 = \frac{a + ((N+1)^{n-1} - 2)\pi}{(N+1)^{n-1}}.$$

Thus

$$A_N(a) = \max \left\{ \frac{a + 2((N+1)^{n-1} - 1)\pi}{(N+1)^{n-1}}, (N+1)\pi \right\},$$

and the proposition follows. $\square$

3.3.2 Embeddings of ellipsoids

We will concentrate on embedding ellipsoids $E^{2n}(\pi, \dots, \pi, a)$ with a very large.

3.3.2.1 Embedding ellipsoids into cubes Studying embeddings $E^{2n}(\pi, \dots, \pi, a) \hookrightarrow C^{2n}(A)$ of skinny ellipsoids into minimal cubes, we face the problem of filling the fibers $\square^{n-1}(A) \times \square^{n-1}(1)$ of the cube by many small fibers $\gamma\triangle^{n-1}(\pi) \times \square^{n-1}(1)$ of the ellipsoid. Forget about the irrelevant y-factors. Since a is very large, γ decreases very slowly. We are thus essentially left with the problem of filling $n-1$-cubes by equal $n-1$-simplices. This is trivial for $n-1=1$ and $n-1=2$, but impossible for $n-1 \geq 3$. Indeed, only 2^{m-1} m-simplices $\triangle^m(\pi)$ fit into $\square^m(\pi)$, whence we only get

$$\lim_{a\to\infty} \frac{|E^{2n}(\pi, \dots, \pi, a)|}{|C^{2n}(s^{2n}_{EC}(a))|} \geq \frac{2^{n-2}}{(n-1)!}. \tag{26}$$

We describe now the embedding process for $n-1=2$ in more detail (cf. Figure 22). We first fill almost half of the "first column" of the cube

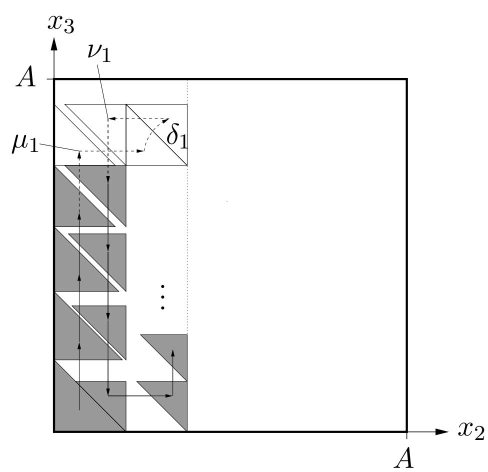

Figure 22: Filling the cube fibres by the ellipsoid fibres

fiber, move the ellipsoid fibre out of this first column (μ_1), deform it to its complementary fiber (δ_1), move this fiber back to the first column (ν_1), and fill almost all of the remaining room in the first column. We then pass to the second column and proceed as before. The deformations δ_i are performed by applying 2-dimensional maps to both symplectic directions

of the ellipsoid fibers (see Figure 25 in 3.3.2.2 and the text belonging to it for more details). In order to guarantee that different stairs do not intersect, we arrange the stairs arising from folding in such a way that the z_1-projections of "upward-stairs" lie in $\{0 < y_1 < 1/2\}$ while the z_1-projections of "downward-stairs" lie in $\{1/2 < y_1 < 1\}$, and we arrange the stairs arising from moving in such a way that the z_1-projections of the μ_i- respectively ν_i-stairs lie in $\{0 < y_1 < 1/4\}$ respectively $\{1/4 < y_1 < 1/2\}$ if i is odd and in $\{1/2 < y_1 < 3/4\}$ respectively $\{3/4 < y_1 < 1\}$ if i is even (cf. Figure 7). The x_1-intervals used for folding respectively moving will then be double respectively four times as large as usual, but this will not affect (26).

Remark. We will prove in subsection 4.1 that the left hand side of (26) is 1 for any n. $\diamond$

3.3.2.2 Embedding ellipsoids into balls

If we try to fill the fibers $\triangle^{n-1}(A) \times \square^{n-1}(1)$ of a ball by many small fibers $\gamma\triangle^{n-1}(\pi) \times \square^{n-1}(1)$ of a skinny ellipsoid, we end up with a result for $s_{EB}^{2n}(a)$ as in (26). In the problem of embedding a skinny ellipsoid into a minimal ball, however, both the fibers of the ellipsoid and the fibers of the ball are balls. This may be used to prove

Proposition 3.13 *For any n,*

$$\lim_{a\to\infty} \frac{|E^{2n}(\pi,\dots,\pi,a)|}{|B^{2n}(s_{EB}^{2n}(a))|} = 1.$$

Proof. The idea of the proof is very simple: Instead of packing a large simplex by small simplices, we will leave the simplices alone and pack the cubes by small cubes, a trivial problem.

So pick a very large $l \in \mathbb{N}$, write

$$P_i = B^{2n}(A) \cap \left\{ \frac{(i-1)A}{l} < x_1 < \frac{iA}{l} \right\}, \qquad 1 \le i \le l,$$

and set

$$k_1 = \frac{A - A/l}{\pi},$$

where A is again a parameter which will be fixed later on. After applying the diagonal map $\operatorname{diag}[k_1,\dots,k_1,1/k_1,\dots,1/k_1]$ to the fibers, the ellipsoid

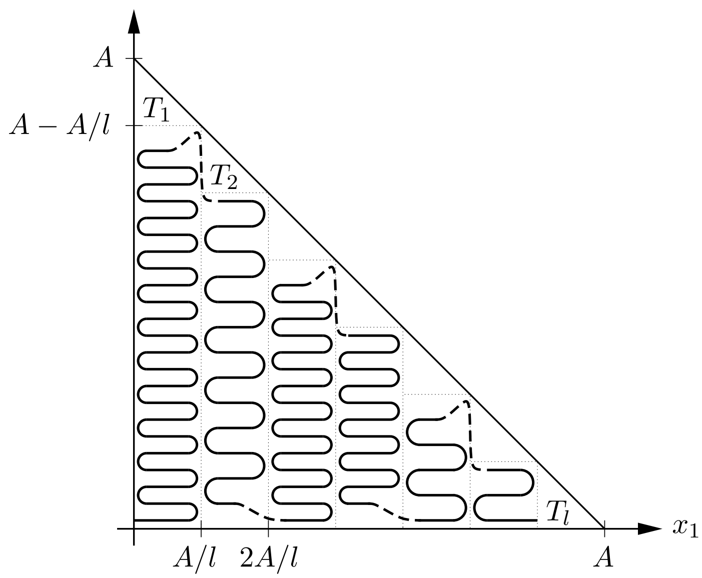

Figure 23: Embedding a skinny ellipsoid into a ball

is contained in $\square(a,1) \times \triangle^{n-1}(k_1\pi) \times \square^{n-1}(1/k_1)$. We will embed some part $\square(b_1,1) \times \triangle^{n-1}(k_1\pi) \times \square^{n-1}(1/k_1)$ of this set into P_1 by fixing the simplices and moving the cubes along the y_i-directions ($2 \le i \le n$) (see Figure 23 and Figure 24).

We want to fill as much of $\square^{n-1}(1)$ by cubes $\square^{n-1}(1/k_1)$ as possible. However, in order to use also the space in P_2 optimally, we will have to deform the ellipsoid fibers before passing to P_2, and for this we will have to use some space in $\square^{n-1}(1)$. Assume that we fold N_1' times in each z_1-z_2-layer and by this embed $\square(b_1',1) \times \triangle^{n-1}(k_1\pi) \times \square^{n-1}(1/k_1)$ into P_1. The maximal ellipsoid fiber over P_2 will then be

$$\left(1 - \frac{b_1'}{a}\right) \triangle^{n-1}(k_1\pi) \times \square^{n-1}\left(\frac{1}{k_1}\right).$$

We want to deform this fiber to a fiber

$$\left(1 - \frac{b_1'}{a}\right) \triangle^{n-1}(k_2'\pi) \times \square^{n-1}\left(\frac{1}{k_2'}\right)$$

fitting into the minimal ball fiber $\triangle^{n-1}(A - 2A/l) \times \square^{n-1}(1)$ over P_2. We thus define k_2' by $(1 - b_1'/a)k_2'\pi = A - 2A/l$. As we shall see below, the

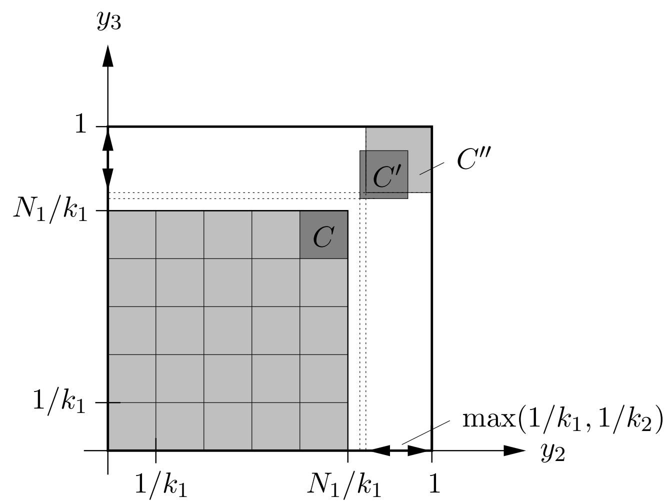

Figure 24: Filling the y-factor of the fibers

appropriate ellipsoid fiber deformation can then be achieved in $\square^{n-1}(1) \setminus \square^{n-1}(1 - \max(1/k_1, 1/k_2))$.

The optimal choice of N_1' and k_2' is the solution of the system

$$\left.\begin{array}{rcl} N_1 & = & \max\left\{N \in \mathbb{N} \,\middle|\, N \text{ even}, \frac{N}{k_1} < 1 - \max\left(\frac{1}{k_1}, \frac{1}{k_2}\right)\right\} \\ k_2\pi & = & \left(A - \frac{2A}{l}\right) \Big/ \left(1 - \frac{b_1(N)}{a}\right) \end{array}\right\}.$$

By folding N_1 times in each z_1-z_2-layer we fill nearly all of $\square^{n-1}(1-\max(1/k_1, 1/k_2))$ and indeed stay away from $\square^{n-1}(1)\setminus\square^{n-1}(1-\max(1/k_1, 1/k_2))$ (cf. Figure 24).

The deformation of the ellipsoid fibres is achieved as follows: We first move the cube C along all y_i-directions, $i \geq 2$, by $1 - \max(1/k_1, 1/k_2) - (N_1 - 1)/k_1 - \epsilon$ for some $\epsilon \in \,]0, 1 - \max(1/k_1, 1/k_2) - N_1/k_1[$. This can be done whenever $A/l > n\pi$. We then deform the translate C' to C''. This deformation is the restriction to $(1 - b_1/a)\triangle^{n-1}(k_1\pi) \times \square^{n-1}(1)$ of a product of $n - 1$ two-dimensional symplectic maps α_i which are explained in Figure 25: On $y_i \leq N_1/k_1$, α_i is the identity, and on $y_i \geq 1 - 1/k_2 - \epsilon$ it is an affine map with linear part

$$(x_i, y_i) \mapsto \left(\frac{k_2}{k_1}x_i, \frac{k_1}{k_2}y_i\right).$$

Assume that we can choose A such that proceeding in this way, we successively fill a large part of all the P_i, $1 \leq i \leq l - 1$, and leave P_l

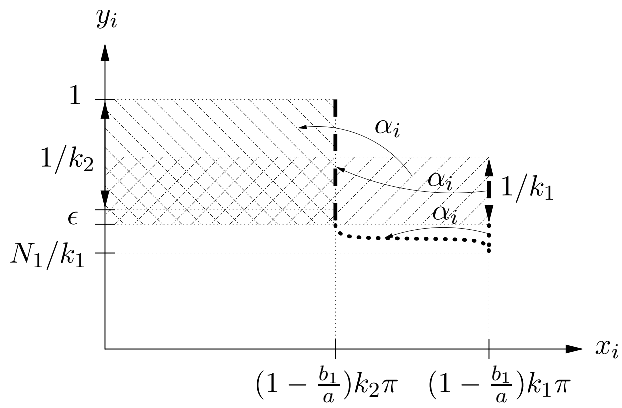

Figure 25: Rescaling the fibers

untouched, i.e. the embedding process ends exactly when passing from P_{l-1} to P_l (cf. Figure 23). The process is then described by the equations for the pairs (N_i, k_{i+1}), $1 \le i \le l-2$,

$$\left.\begin{array}{rcl} N_i & = & \max\left\{N \in \mathbb{N} \,\middle|\, N \text{ even}, \frac{N}{k_i} < 1 - \max\left(\frac{1}{k_i}, \frac{1}{k_{i+1}}\right)\right\} \\ k_{i+1}\pi & = & \left(A - \frac{(i+1)A}{l}\right) \Big/ \left(1 - \frac{\sum_{j=1}^{i-1} b_j(N_j) + b_i(N)}{a}\right) \end{array}\right\}, \tag{27}$$

where $b_j(N_j)$ is the x_1-length of the part embedded into P_j, and by

$$N_{l-1} = \max\{n \in \mathbb{N} \mid N \text{ even}, \, N < k_{l-1}\}.$$

We finally observe that, in reality, the system (27) splits. Indeed, the second line in (27) readily implies that $k_i < 2k_{i+1}$ whenever $i \le l-2$. Thus, the first line in (27) reads $N_i = \max\{N \in \mathbb{N} \mid N \text{ even}, \, N/k_i < 1 - 1/k_i\}$, and the embedding process is described by

$$N_i = \max\{N \in 2\mathbb{N} \mid N < k_i - 1\} \tag{28.1}$$

$$k_{i+1}\pi = \left(A - \frac{(i+1)A}{l}\right) \Bigg/ \left(1 - \frac{\sum_{j=1}^{i} b_j(N_j)}{a}\right) \tag{28.2}$$

$$N_{l-1} = \max\{N \in 2\mathbb{N} \mid N < k_{l-1}\}. \tag{28.3}$$

We now argue that such an A indeed exists, and that it is the minimal A for which the above embedding process succeeds.

Observe first that such a minimal A, which we denote by A_0, indeed exists, for clearly, if A was chosen very large, the embedding process will

end at some P_i with $i < l-1$, and if A was chosen very small, it won't succeed at all.

Suppose now that the embedding process for A_0 ends before passing from P_{l-1} to P_l. Pick $A' < A_0$ and write k_i and N_i respectively k_i' and N_i' for the embedding parameters belonging to A_0 respectively A'. If $A_0 - A'$ is small, $k_1 - k_1'$ is small too; thus, by (28.1), $N_1 = N_1'$ whenever $A_0 - A'$ is small enough. But then, $b_1(N_1) - b_1'(N_1)$ is small, whence (28.2) shows that $k_2 - k_2'$ is small. Arguing by induction, we assume that $N_j = N_j'$ and that $b_j(N_j) - b_j'(N_j)$ and $k_{j+1} - k_{j+1}'$ are small for $j \le i$. Then, by (28.1) or (28.3), and after choosing $A_0 - A'$ even smaller if necessary, we may assume that $N_{i+1} = N_{i+1}'$. If $i+2 \le l-1$, $b_{j+1}(N_{j+1}) - b_{j+1}'(N_{j+1})$ is then small too, whence (28.2) shows that $k_{i+2} - k_{i+2}'$ is small.

We hence may assume that all differences $b_i - b_i'$ are arbitrarily small. But then the embedding process for A' will succeed as well, a contradiction.

Recall that $A_0 = A_0(a,l)$ still depends on l. The best embedding result provided by the above procedure is thus

$$s_{EB}^{2n}(a) = \min_{l \in \mathbb{N}} \{A_0(a,l)\}.$$

Set

$$q(a,l) = 1 - \frac{|E^{2n}(\pi,\dots,\pi,a)|}{|B^{2n}(A_0(a,l))|}$$

and

$$q(a) = 1 - \frac{|E^{2n}(\pi,\dots,\pi,a)|}{|B^{2n}(s_{EB}^{2n}(a))|}.$$

In order to prove the proposition, we have to show that

$$\lim_{a \to \infty} q(a) = 0. \tag{29}$$

Given any a and l, the region in $B^{2n}(A_0(a,l))$ which is not covered by the image of $E^{2n}(\pi,\dots,\pi,a)$ is the disjoint union of four types of regions $R_h(a,l)$, $1 \le h \le 4$.

$R_1(a,l)$ is the union of the "triangles" $T_i(a,l)$ (see Figure 23).

$R_2(a,l)$ is the space needed for folding (see Figure 28).

$R_3(a,l)$ is the union of the space needed to deform the ellipsoid fibers and the space caused by the fact that the N_i have to be integers (see Figure 24).

$R_4(a, l)$ is the image of the difference set of the embedded set and $E^{2n}(\pi, \dots, \pi, a)$ (see Figure 26).

Detailed descriptions of these sets are given below.

Let $\epsilon > 0$ be small. We will find a_ϵ and l_ϵ such that

$$\frac{|R_h(a, l_\epsilon)|}{|B^{2n}(A_0(a, l_\epsilon))|} < \epsilon \quad \text{for all } a \geq a_\epsilon, \tag{30.h}$$

$1 \leq h \leq 4$. Since the sets $R_h(a, l)$ are disjoint and $q(a) \leq q(a, l)$, (30.h), $1 \leq h \leq 4$, imply (29).

Set $R_{h,i}(a, l) = R_h(a, l) \cap P_i(a, l)$. We first of all observe that the ratio $|R_1(a, l)|/|B^{2n}(A_0(a, l))|$ depends only on l and can be made arbitrarily small by taking l large. We thus find l_1 such that

$$\frac{|R_1(a, l)|}{|B^{2n}(A_0(a, l))|} < \epsilon \quad \text{for all } a \text{ and } l \geq l_1.$$

Moreover, notice that given $\zeta > 0$ we can choose l_1 such that for all a and $l \geq l_1$

$$\frac{|R_{1,i}(a, l)|}{|P_i(a, l)|} < \zeta \quad \text{whenever } i \text{ is not too near to } l - 1. \tag{31}$$

Here and in the sequel, "i too near to $l-1$" stands for "$1 - i/(l-1)$ smaller than a constant which can be made arbitrarily small by taking first l and then also a large".

Next, our construction clearly shows that given ζ as above and l being fixed we may find a_1 such that for $a \geq a_1$ and for all $i \in \{1, \dots, l-1\}$

$$\frac{|R_{2,i}(a, l)|}{|P_i(a, l)|} < \zeta \quad \text{and} \quad \frac{|R_{3,i}(a, l)|}{|P_i(a, l)|} < \zeta. \tag{32}$$

In particular, given any $l_\epsilon \geq l_1$, we find a_ϵ such that (30.1), (30.2) and (30.3) hold true.

Recall that the embedding $\varphi_{a,l} \colon E^{2n}(\pi, \dots, \pi, a) \hookrightarrow B^{2n}(A_0(a, l))$ is defined on a larger domain with piecewise constant fibres. Set

$$\begin{aligned} X_i(a, l) &= \varphi_{a,l}^{-1}(P_i(a, l)), \\ Y_i(a, l) &= X_i(a, l) \setminus E^{2n}(\pi, \dots, \pi, a), \\ Z_i(a, l) &= X_i(a, l) \cap E^{2n}(\pi, \dots, \pi, a) \end{aligned}$$

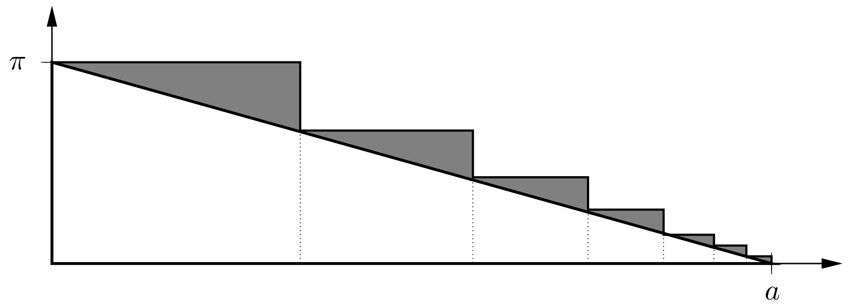

Figure 26: $Y(a,8) \subset X(a,8)$

and $X(a,l) = \coprod_{i=1}^{l-1} X_i(a,l)$, $Y(a,l) = \coprod_{i=1}^{l-1} Y_i(a,l)$, $Z(a,l) = \coprod_{i=1}^{l-1} Z_i(a,l)$ (cf. Figure 26), and recall that we denoted the u-width of $X_i(a,l)$ by $b_i(a,l)$. Assume now that ζ is small. Then (31) and (32) show that for $a \geq a_\epsilon$ and i not too near to $l_\epsilon - 1$, $|X_i(a,l_\epsilon)|/|P_i(a,l_\epsilon)|$ is near to 1. Thus, a simple volume comparison shows that if l_ϵ is large, $b_i(a,l_\epsilon)/a$ and hence also $|R_{4,i}(a,l_\epsilon)|/|P_i(a,l_\epsilon)| = |Y_i(a,l_\epsilon)|/|P_i(a,l_\epsilon)|$ is small for these a and i. In particular, we may choose l_ϵ and a_ϵ such that (30.4) holds true too.

This completes the proof of Proposition 3.13. For later purposes, we state that given $\zeta > 0$, we may find l_0 and a_0 such that for all $a \geq a_0$ and i not too near to $l_0 - 1$

$$\frac{|R_{h,i}(a,l_0)|}{|P_i(a,l_0)|} < \zeta, \qquad 1 \leq h \leq 4. \tag{33}$$

□

The above proof gives no information about the convergence speed in (29). The remainder of this paragraph is devoted to the proof of

Proposition 3.14 *Given $\epsilon > 0$ there is a constant $C(n,\epsilon)$ such that for all a*

$$1 - \frac{|E^{2n}(\pi,\ldots,\pi,a)|}{|B^{2n}(s_{EB}^{2n}(a))|} < C(n,\epsilon)a^{-\frac{1}{2n}+\epsilon}.$$

Proof. The proposition follows from the existence of a pair (a_0,l_0) such that for $a \in I_k(a_0) = [4^{kn}a_0, 4^{(k+1)n}a_0[$, $k \in \mathbb{N}_0$,

$$(2-\epsilon)q(4^n a, 2^{k+1}l_0) < q(a, 2^k l_0). \tag{34}$$

Indeed, choose $C(n,\epsilon)$ so large that $C(n,\epsilon)a^{-\frac{1}{2n}+\epsilon} > q(a)$ for $a < a_0$ and

$$C(n,\epsilon)a^{-\frac{1}{2n}} > q(a,l_0) \quad \text{ for } a \in I_0(a_0). \tag{35}$$

Then, if $a \in I_k(a_0)$ for some $k \in \mathbb{N}$,

$$\begin{aligned} q(a) \;\leq\; q(a, 2^k l_0) \;&\overset{(34)}{<}\; (2-\epsilon)^{-k} q\left(\frac{a}{4^{kn}}, l_0\right) \\ &\overset{(35)}{<}\; (2-\epsilon)^{-k} C(n,\epsilon) 2^k a^{-\epsilon} a^{-\frac{1}{2n}+\epsilon} \\ &\leq\; (2-\epsilon)^{-k} C(n,\epsilon) 2^k 4^{-\epsilon k n} a_0^{-\epsilon} a^{-\frac{1}{2n}+\epsilon} \\ &<\; (2-\epsilon)^{-k} 2^k 4^{-\epsilon k n} C(n,\epsilon) a^{-\frac{1}{2n}+\epsilon} \\ &<\; C(n,\epsilon) a^{-\frac{1}{2n}+\epsilon}. \end{aligned}$$

So let's prove (34). Fix (a_0, l_0) and $\hat{a} \in I_0(a_0)$ and set $a_k = 4^{kn} a_0$, $\hat{a}_k = 4^{kn}\hat{a}$, $l_k = 2^k l_0$ and

$$\rho_k = \frac{A_0(\hat{a}_{k+1}, l_{k+1})}{A_0(\hat{a}_k, l_k)},$$

$k \in \mathbb{N}_0$. Given a specified subset $S(a,l)$ of $B^{2n}(A_0(a,l))$ and a parameter $p(a,l)$ belonging to the embedding $\varphi_{a,l}\colon E^{2n}(\pi,\dots,\pi,a) \hookrightarrow B^{2n}(A_0(a,l))$, we write ${}_kS$ and ${}_kp$ instead of $S(\hat{a}_k, l_k)$ and $p(\hat{a}_k, l_k)$. Moreover, we write ${}_kS'$ for the rescaled subset $\frac{1}{\rho_k} S(\hat{a}_{k+1}, l_{k+1})$ of $\frac{1}{\rho_k} B^{2n}(A_0(\hat{a}_{k+1}, l_{k+1}))$ and ${}_kp'$ for the parameter belonging to the rescaled embedding $\frac{1}{\rho_k} E^{2n}(\pi,\dots,\pi,\hat{a}_{k+1}) \hookrightarrow \frac{1}{\rho_k} B^{2n}(A_0(\hat{a}_{k+1}, l_{k+1}))$. Finally, write ρ, S, S', p, p' instead of ρ_0, ${}_0S$, ${}_0S'$, ${}_0p$, ${}_0p'$, set $E = E^{2n}(\pi,\dots,\pi,\hat{a})$, $E' = \frac{1}{\rho} E^{2n}(\pi,\dots,\pi,\hat{a}_1)$ and $B = B^{2n}(A_0(\hat{a}, l_0))$, and observe that $B = B'$.

We claim that we can find (a_0, l_0) such that for all $k \in \mathbb{N}_0$, $\hat{a}_k \in I_k(a_0)$ and i not too near to $l_k - 1$

$$(4-\epsilon)|{}_kR'_{h,2i(-1)}| < |{}_kR_{h,i}|, \tag{36.h.k}$$

$1 \leq h \leq 4$. We will first prove (36.h.0) and will then check that the conditions valid for $(\hat{a}, l_0)$ which allowed us to conclude (36.h.0) are also valid for $(\hat{a}_k, l_k)$ provided that (36.h.m) holds true for $m \leq k-1$. Arguing by induction, we thus see that (36.h.k) holds true for all $k \in \mathbb{N}_0$.

Set $\epsilon_1 = \epsilon/16$ and observe that for all $k \in \mathbb{N}_0$ and i not too near to $l_k - 1$

$$|{}_kP'_{2i-1}| > |{}_kP'_{2i}| > \left(\frac{1}{2} - \epsilon_1\right) |{}_kP_i|. \tag{37}$$

We conclude that for $k \in \mathbb{N}_0$, $\hat{a}_k \in I_k(a_0)$ and i not too near to $l_k - 1$

$$\left(2 - \frac{3\epsilon}{4}\right) \frac{|R_{h,2i(-1)}(\hat{a}_{k+1}, l_{k+1})|}{|P_{2i(-1)}(\hat{a}_{k+1}, l_{k+1})|} < \frac{|R_{h,i}(\hat{a}_k, l_k)|}{|P_i(\hat{a}_k, l_k)|}, \qquad \text{(38.h.k)}$$

$1 \le h \le 4$. In particular, there is (a_0, l_0) such that for all $\hat{a} \in I_0(a_0)$,

$$(2-\epsilon)\frac{|R_h(\hat{a}_{k+1}, l_{k+1})|}{|B^{2n}(A_0(\hat{a}_{k+1}, l_{k+1}))|} < \frac{|R_h(\hat{a}_k, l_k)|}{|B^{2n}(A_0(\hat{a}_k, l_k))|},$$

$1 \le h \le 4$. Since $R_h(a,l)$ are disjoint, this implies (34).

(R1) Let $R_1(a,l) = \coprod_{i=1}^{l} T_i(a,l)$ be the union of the "triangles" $T_i(a,l) \subset B^{2n}(A_0(a,l))$ (see Figure 23). $R'_{1,2i(-1)}$ is a subset of $R_{1,i}$, and $|R_{1,i}|/|R'_{1,2i(-1)}| =$

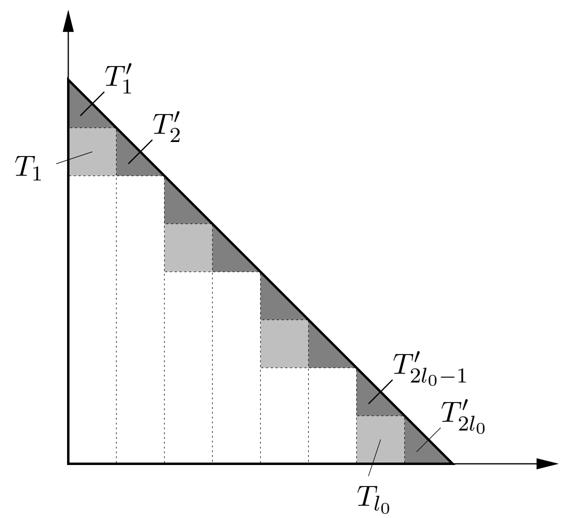

Figure 27: R_1 and R'_1

$|T_i|/|T'_{2i(-1)}|$ depends only on l_0 (see Figure 27). Clearly, $4 - |T_i|/|T'_{2i(-1)}|$ is small if $|T_i|/|P_i|$ is small enough. By taking l_0 large, we may make $|T_i|/|P_i|$ arbitrarily small for i not too near to $l_0 - 1$. Thus, (36.1.0) holds true whenever l_0 is large enough. Observe finally that (36.1.0) implies (36.1.k), $k \in \mathbb{N}$.

(R2) Recall that the x_1-length of the space needed for folding equals the fiber capacity at the place where we fold. The staircases needed for folding

are thus contained in $R_2(a,l)=\coprod_{i=1}^{l-1} R_{2,i}(a,l)$, where $R_{2,i}(a,l)$ equals

$$Q_i(a,l) \setminus \left\{ \frac{(i-1)A}{l} + \pi\left(1 - \frac{\sum_{j=1}^{i-1} b_j}{a}\right) < x_1 < \frac{iA}{l} - \pi\left(1 - \frac{\sum_{j=1}^{i-1} b_j}{a}\right) \right\}.$$

Here, we put

$$Q_i(a,l) = P_i(a,l) \setminus T_i(a,l).$$

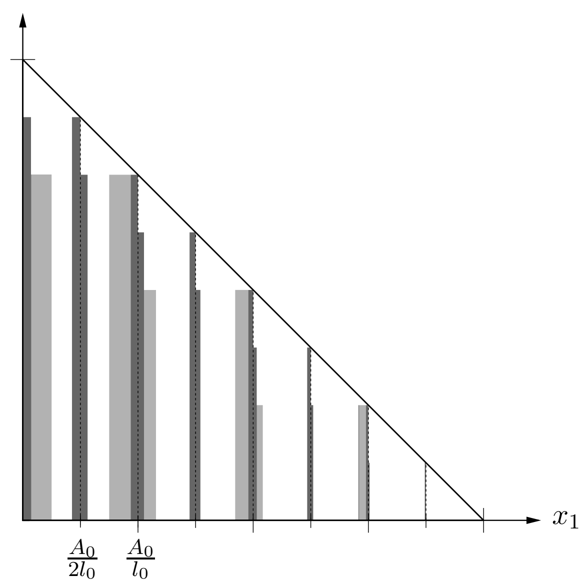

Figure 28: R_2 and R_2'

Observe that for i not too near to $l_k - 1$, $|{}_kQ'_{2i-1} \cap {}_kT_i|/|{}_kQ'_{2i-1}| \to 0$ as $l_k \to \infty$ (cf. Figure 27). Hence, also $|{}_kR'_{2,2i-1} \cap {}_kT_i|/|{}_kR'_{2,2i-1}| \to 0$ as $l_k \to \infty$. We may thus neglect ${}_kR'_{2,2i-1} \cap {}_kT_i$ and prove (36.2.k) with ${}_kR'_{2,2i-1}$ replaced by ${}_kR'_{2,2i-1} \setminus {}_kT_i$ (which we denote again by ${}_kR'_{2,2i-1}$).

If $u_i = \sum_{j=1}^{i-1} b_j$ respectively $u_i' = \sum_{j=1}^{i-1} b_j'$ is the x_1-coordinate at which the image of E respectively E' enters P_i, then the volume embedded into $\coprod_{j=1}^{i-1} P_j$ is

$$\frac{\pi^{n-1}}{\hat{a}^{n-1} n!}\left[\hat{a}^n - (\hat{a} - u_i)^n\right] \quad \text{resp.} \quad \frac{\pi^{n-1}}{\hat{a}_1^{n-1} n!}\left[\left(\frac{\hat{a}_1}{\rho}\right)^n - \left(\frac{\hat{a}_1}{\rho} - u_i'\right)^n\right], \quad (39)$$

and the fiber capacity at u_i respectively u_i' is

$$c_i = \frac{\pi}{\hat{a}}(\hat{a} - u_i) \quad \text{resp.} \quad c_i' = \frac{\pi}{\hat{a}_1}\left(\frac{\hat{a}_1}{\rho} - u_i'\right). \tag{40}$$

Thus, $c_1 = \rho c_1'$. We claim that

$$c_i > (1 - \epsilon_1)\rho c_{2i(-1)}' \qquad \text{whenever } \hat{a} \text{ is large enough and } i \text{ is not too near to } l_0 - 1. \tag{41}$$

Since $c_{2i-1}' > c_{2i}'$, it suffices to show that

$$c_i > (1 - \epsilon_1)\rho c_{2i-1}' \qquad \text{for } \hat{a} \text{ large enough and } i \text{ not too near to } l_0 - 1. \tag{41'}$$

So assume that there is an i violating the inequality in (41') and set

$$i_0 = \min\{1 \le i \le l_0 - 1 \,|\, c_i \le (1 - \epsilon_1)\rho c_{2i-1}'\}.$$

Let $\zeta > 0$ be so small that

$$\zeta < \epsilon_1 \tag{42}$$

and set

$$z_i(a, l) = \frac{|Z_i(a, l)|}{|P_i(a, l)|} \quad \text{and} \quad z(a, l) = \frac{|Z(a, l)|}{|B^{2n}(A_0(a, l))|}.$$

By the definition of ρ, z and z',

$$\rho^n = 4^n \frac{z}{z'}. \tag{43}$$

By (33), for any large enough l_0 there is a_0 such that for all $\hat{a} \in I_0(a_0)$ and i not too near to $l_0 - 1$

$$z_i > 1 - \zeta. \tag{44}$$

We have seen in (R_1) that for all $i \in \{1, \dots, l_0\}$

$$|R_{1,2i(-1)}'| < |R_{1,i}|. \tag{45}$$

Moreover, if ζ is small enough, we clearly have that for i not too near to $l_0 - 1$

$$c_i > c_{2i(-1)}'. \tag{46}$$

This implies that for these i

$$|R'_{2,2i(-1)}| < |R_{2,i}|. \tag{47}$$

We now assume that a_0 is so large compared to l_0 that

$$A_0(a_0, l_0) > 12 l_0 \pi. \tag{48}$$

Then, $A_0(\hat{a}, l_0) > 12 l_0 \pi > 12 l_0 c_i$, i.e.

$$\frac{A_0(\hat{a}, l_0)}{l_0} > 12 c_i, \qquad 1 \leq i \leq l_0 - 1. \tag{49}$$

$$|R'_{3,2i(-1)}| < |R_{3,i}| \tag{50}$$

now follows from (46) in the same way as (73) will follow from (41). Finally, for ζ small enough and i not too near to $l_0 - 1$ we clearly have that

$$|R'_{4,2i(-1)}| < |R_{4,i}|. \tag{51}$$

We conclude from (45), (47), (50) and (51) and (37) that

$$\frac{|R'_{h,2i(-1)}|}{|P'_{2i(-1)}|} < 3 \frac{|R_{h,i}|}{|P_i|}, \qquad 1 \leq h \leq 4.$$

This shows that

$$z'_i > 1 - 3\zeta. \tag{52}$$

Set

$$z_{<i} = \frac{|\coprod_{j=1}^{i-1} Z_j|}{|\coprod_{j=1}^{i-1} P_j|} \quad \text{and} \quad z'_{<i} = \frac{|\coprod_{j=1}^{i-1} Z'_j|}{|\coprod_{j=1}^{i-1} P'_j|}. \tag{53}$$

By (44) and (52), we may assume that for all $i \in \{1, \dots, l_0 - 1\}$

$$z_{<i} > 1 - \zeta \quad \text{and} \quad z'_{<i} > 1 - 3\zeta.$$

In particular,

$$z > 1 - \zeta \quad \text{and} \quad z' > 1 - 3\zeta \tag{54}$$

and

$$z_{<i_0} > 1 - \zeta \quad \text{and} \quad z'_{<i_0} > 1 - 3\zeta. \tag{55}$$

Comparing the two volumes embedded into $\coprod_{j=1}^{i_0-1} P_j$, we get from (39) that

$$z'_{<i_0} \frac{\pi^{n-1}}{\hat{a}^{n-1} n!} \left[\hat{a}^n - (\hat{a} - u_{i_0})^n\right] = z_{<i_0} \frac{\pi^{n-1}}{\hat{a}_1^{n-1} n!} \left[\left(\frac{\hat{a}_1}{\rho}\right)^n - \left(\frac{\hat{a}_1}{\rho} - u'_{2i_0-1}\right)^n\right]. \tag{56}$$

By (40), $c_{i_0} \le (1-\epsilon_1)\rho c'_{2i_0-1}$ translates to

$$u'_{2i_0-1} \le \frac{4^n}{(1-\epsilon_1)\rho}(u_{i_0} - \epsilon_1 \hat{a}). \tag{57}$$

Plugging (57) into (56), we find

$$\left(z_{<i_0}\left(\frac{4}{\rho}\right)^n - z'_{<i_0}\right)\hat{a}^n \ge \left(z_{<i_0}\left(\frac{4}{\rho(1-\epsilon_1)}\right)^n - z'_{<i_0}\right)(\hat{a} - u_{i_0})^n,$$

and using (43) and dividing by $z_{<i_0}$ we get

$$\left(\frac{z'}{z} - \frac{z'_{<i_0}}{z_{<i_0}}\right)\hat{a}^n \ge \left(\frac{z'}{z}\frac{1}{(1-\epsilon_1)^n} - \frac{z'_{<i_0}}{z_{<i_0}}\right)(\hat{a} - u_{i_0})^n. \tag{58}$$

By (54) and (55), $|1 - z'/z|$ and $|1 - z'_{<i_0}/z_{<i_0}|$ can be made arbitrarily small by taking ζ small. (58) thus shows that for ζ small enough, $1 - u_{i_0}/\hat{a}$ must be small, i.e. i_0 must be near to $l_0 - 1$. This concludes the proof of (41').

Putting everything together, we see that l_0 and a_0 may be chosen such that for i not too near to $l_0 - 1$

$$\begin{aligned}
|R_{2,i}| \overset{(41)}{>} (1-\epsilon_1)\rho|R'_{2,2i(-1)}| &\overset{(43),(54)}{>} (1-\epsilon_1)4\sqrt[n]{1-\zeta}\,|R'_{2,2i(-1)}| \\
&\overset{(42)}{>} 4(1-\epsilon_1)^2|R'_{2,2i(-1)}| \\
&> (4-\epsilon)|R'_{2,2i(-1)}|.
\end{aligned}$$

This proves (36.2.0).

Suppose now that (36.h.m), $1 \le h \le 4$, and hence also (38.h.m) hold true for $m \le k-1$. (38.h.m) and (44) imply that for i not too near to $l_k - 1$

$$_k z_i > 1 - \zeta. \tag{59}$$

The reasoning which implied (46) thus also shows that for i as in (46)

$$_k c_{2^k i} > {_k c'_{2^{k-1} i}}. \tag{60}$$

Since l_0 is large and ζ is small, $_k c_{2^{k-1} i} - {_k c_{2^k i}}$ is small. We thus see that for i not too near to $l_0 - 1$

$$_k c_i > {_k c'_{2i(-1)}} \tag{61}$$

almost holds true, and hence also

$$|_k R'_{2,2i(-1)}|| < |_k R_{2,i}| \tag{62}$$

almost holds true. Next, observe that (44) and (59) imply that $A_0(a_k, l_k)/A_0(a_0, l_0)$ is near to 4^k. This and (48) show that

$$A_0(a_k, l_k) > 12 l_k \pi, \tag{63}$$

and in the same way as we derived (50) from (46) and (49) we may derive from (61) and (63) that

$$|_k R'_{3,2i(-1)}| < |_k R_{3,i}| \tag{64}$$

almost holds true. Finally, by (59), we also have that for i not too near to $l_k - 1$

$$|_k R'_{4,2i(-1)}| < |_k R_{4,i}| \,. \tag{65}$$

We infer from (37), (62), (64) and (65) that

$$\frac{|_k R'_{h,2i(-1)}|}{|_k P'_{2i(-1)}|} < 3 \frac{|_k R_{h,i}|}{|_k P_i|}, \qquad 1 \leq h \leq 4,$$

i.e.

$$_k y'_i > 1 - 3\zeta.$$

Proceeding exactly as in the case $k = 0$ we thus get that for i not too near to $l_k - 1$

$$_k c_i > (1 - \epsilon_1) \rho_k \, {_k c'_{2i(-1)}}, \tag{66}$$

from which (36.2.k) follows in the same way as for $k = 0$.

(R3) Set

$$D_i(a,l) = \square^{n-1}(1) \setminus \square^{n-1}(N_i k_i)$$

and

$$W_i(a,l) = \left] \sum_{j=1}^{i-1} b_j(a,l), \sum_{j=1}^{i} b_j(a,l) \right[\times]0,1[\times \left(1 - \frac{\sum_{j=1}^{i-1} b_j(a,l)}{a} \right) \triangle^{n-1}(\pi), \tag{67}$$

$1 \leq i \leq l-1$. Moreover, let C_i be the cube in the y-factor of the fibers which will be deformed and let K_i be the extra space in P_i needed to move C_i along the y_j-directions, $j \geq 2$. Then,

$$R_3(a,l) = \varphi_{a,l}\left(\coprod_{i=1}^{l-1} W_i(a,l) \right) \times D_i(a,l) \cup \coprod_{i=1}^{l-2} K_i.$$

We first of all observe that $K_i \subset \varphi_{a,l}(W_i(a,l)) \times C_i$ and that $|C_i|/|D_i(a,l)|$ is small for i not too near to $l-1$ and a large, since then $k_i(a,l)$ is large. We thus may forget about the K_i. Next, as in (R_2), notice that for i not too near to $l_k - 1$,

$$|{}_kR'_{3,2i-1} \cap {}_kT_i| / |{}_kR'_{3,2i-1}| \to 0 \quad \text{as} \quad l_k \to \infty,$$

whence we may neglect ${}_kR'_{3,2i-1} \cap {}_kT_i$ and prove (36.3.k) with ${}_kR'_{3,2i-1}$ replaced by ${}_kR'_{3,2i-1} \setminus {}_kT_i$ (which we denote again by ${}_kR'_{3,2i-1}$).

By (28.1),

$$N_i(a,l) = \begin{cases} k_i - 2, & (k_i \text{ even}) \\ k_i - 3, & (k_i \text{ odd}) \end{cases} \qquad \text{for } 1 \leq i \leq l-2. \tag{68}$$

This and Figure 24 show that for these i,

$$\left(1 - \frac{3}{k_i(a,l)} \right)(n-1)\left(1 - \frac{N_i(a,l)}{k_i(a,l)} \right) < |D_i(a,l)| < (n-1)\left(1 - \frac{N_i(a,l)}{k_i(a,l)} \right). \tag{69}$$

Observe now that $c_i k_i = c'_{2i} k'_{2i} < c'_{2i-1} k'_{2i-1}$. Hence, by (41),

$$k'_{2i(-1)} > (1 - \epsilon_1)\rho k_i \tag{70}$$

if i is not too near to $l_0 - 1$. (68) and (70) imply that for these i

$$\frac{1 - N_i/k_i}{1 - N'_{2i(-1)}/k'_{2i(-1)}} > \frac{2}{3}(1-\epsilon_1)\rho. \tag{71}$$

Using again that for i not too near to $l-1$, $k_i(a,l)$ is large whenever a is large, we conclude from (69) and (71) that for a_0 large enough and i not too near to $l_0 - 1$,

$$\frac{|D_i|}{|D'_{2i(-1)}|} > \frac{2}{3}(1-2\epsilon_1)\rho. \tag{72}$$

We conclude that for such a_0 and i

$$|R_{3,i}|/|R'_{3,2i(-1)}| \overset{(49),(72)}{>} 2\frac{5}{6}\frac{2}{3}(1-2\epsilon_1)\rho \;>\; \frac{10}{9}(1-2\epsilon_1)4(1-\epsilon_1) \;>\; 4-\epsilon. \tag{73}$$

This proves (36.3.0).

Suppose again that (36.h.m), $1 \le h \le 4$, holds true for $m \le k-1$. Then (66) implies

$$_k k'_{2i(-1)} > (1-\epsilon_1)\rho_k \, _k k_i$$

if i is not too near to $l_k - 1$, and proceeding as before we obtain (36.3.k).

(R4) Recall that $R_4(a,l) = \varphi_{a,l}(Y(a,l))$ (cf. Figure 26).

To any partition $\bar{Z} = \coprod_{i=1}^{l-1} \bar{Z}_i$ of $E^{2n}(\pi,\dots,\pi,\bar{a})$ looking as in Figure 26 associate the set $X(\bar{Z}) = \coprod X_i(\bar{Z})$ which is obtained from $\bar{Z}$ by replacing each fiber in $\bar{Z}_i$ by the maximal fiber in $\bar{Z}_i$ (see Figure 26). Set $Y_i(\bar{Z}) = X_i(\bar{Z}) \setminus \bar{Z}_i$ and $Y(\bar{Z}) = \coprod Y_i(\bar{Z})$. Clearly, if the partitions $E^{2n}(\pi,\dots,\pi,\bar{a}) = \coprod_{i=1}^{l-1} \bar{Z}_i$ and $E^{2n}(\pi,\dots,\pi,\bar{\bar{a}}) = \coprod_{i=1}^{l-1} \bar{\bar{Z}}_i$ are similar to each other, then

$$\frac{|Y_i(\bar{Z})|}{|\bar{Z}_i|} = \frac{|Y_i(\bar{\bar{Z}})|}{|\bar{\bar{Z}}_i|}. \tag{74}$$

Let $B^{2n}(\bar{A}) = \coprod_{i=1}^{l} \bar{P}_i$ be a partition as in Figure 28 and assume that

$$\frac{|\bar{Z}_i|}{|\bar{P}_i|} > 1-\zeta \quad \text{and} \quad \frac{|\bar{\bar{Z}}_i|}{|\bar{P}_i|} > 1-\zeta \quad \text{for } 1 \le i \le i_0.$$

Clearly, if ζ is small enough and i_0 is large enough, $\bar{Z}$ and $\bar{\bar{Z}}$ are almost similar. (74) thus shows that given i_1 not too large we may find ζ and i_0 such that for $i \leq i_1$

$$\frac{|Y_i(\bar{Z})|}{|\bar{Z}_i|} < (1+\epsilon_1)\frac{|Y_i(\bar{\bar{Z}})|}{|\bar{\bar{Z}}_i|}. \tag{75}$$

Given $\hat{a}_m \in I_m(a_0)$, $m \in \mathbb{N}_0$, and $1 \leq i \leq l_0 - 1$, set

$$Z_i(\hat{a}_m) = \coprod_{j=2^m(i-1)+1}^{2^m i} Z_j(\hat{a}_m, l_m),$$

$Z(\hat{a}_m) = \coprod Z_i(\hat{a}_m)$, $P(Z_i(\hat{a}_m)) = \coprod_{j=2^m(i-1)+1}^{2^m i} P(Z_j(\hat{a}_m, l_m))$ and $z(Z_i(\hat{a}_m)) = |Z_i(\hat{a}_m)|/|P(Z_i(\hat{a}_m))|$. For a_0 large and i as above we clearly have that for all $m \in \mathbb{N}_0$ and $\hat{a}_m \in I_m(a_0)$

$$\frac{\left|\coprod_{j=2^m(i-1)+1}^{2^m i} Y(Z_j(\hat{a}_m, l_m))\right|}{|P(Z_i(\hat{a}_m))|} \leq \frac{|Y_i(\hat{a}, l_0)|}{|P_i(\hat{a}, l_0)|}. \tag{76}$$

Assume now that for some m, i not too near to $l_0 - 1$ and $2^m(i-1)+1 \leq j \leq 2^m$

$$\frac{\left|R_{h,j}(\hat{a}_m, l_m)\right|}{\left|P_j(\hat{a}_m, l_m)\right|} \leq \frac{1}{(2-\epsilon)^m}\frac{|R_{h,i}(\hat{a}, l_0)|}{|P_i(\hat{a}, l_0)|}, \qquad 1 \leq h \leq 3. \tag{77}$$

(76) and (77) in particular imply that for these i

$$z(Z_i(\hat{a}_m)) \geq z_i. \tag{78}$$

(78) and (75) imply that l_0 and a_0 may be chosen such that for all $\hat{a}_m$, $\hat{a}_{m'}$ satisfying (77) and i not too near to $l_0 - 1$

$$\frac{|Y_i(Z(\hat{a}_m))|}{|Z_i(\hat{a}_m)|} < (1+\epsilon_1)\frac{|Y_i(Z(\hat{a}_{m'}))|}{|Z_i(\hat{a}_{m'})|}. \tag{79}$$

Suppose now that (36.h.m), $1 \leq h \leq 4$, holds true for $m \leq k-1$. We then have shown in (R_h), $1 \leq h \leq 3$, that (77) holds true for $m \leq k+1$. (79) thus implies that for i not too near to $l_0 - 1$

$$\frac{|Y_i(Z(\hat{a}_{k+1}))|}{|Z_i(\hat{a}_{k+1})|} < (1+\epsilon_1)\frac{|Y_i(Z(\hat{a}_k))|}{|Z_i(\hat{a}_k)|},$$

and (78) with $m = k$ now shows that for these i

$$\frac{|Y_i(Z(\hat{a}_{k+1}))|}{|P(Z_i(\hat{a}_{k+1}))|} < \frac{1+\epsilon_1}{1-\zeta}\frac{|Y_i(Z(\hat{a}_k))|}{|P(Z_i(\hat{a}_k))|}. \tag{80}$$

Pick ϵ_2 so small that

$$\left(1-\frac{\epsilon}{4}\right)\frac{1+\epsilon_2}{1-\epsilon_2}\frac{1+\epsilon_1}{1-\zeta} < 1. \tag{81}$$

This is possible since

$$\left(1-\frac{\epsilon}{4}\right)\frac{1+\epsilon_1}{1-\zeta} \overset{(42)}{<} \left(1-\frac{\epsilon}{4}\right)\frac{1+\epsilon_1}{1-\epsilon_1} < 1.$$

We will show that l_0 and a_0 can be chosen such that for any $\hat{a}_m$ satisfying (78), i not too near to $l_0 - 1$ and $2^m(i-1)+1 \le j \le 2^m i$

$$(1-\epsilon_2)|Y(Z_i(\hat{a}_m))| < 4^m|Y_j(\hat{a}_m, l_m)| < (1+\epsilon_2)|Y(Z_i(\hat{a}_m))|. \tag{82}$$

The second inequality in (82) with $m = k+1$, (80), the first inequality in (82) with $m = k$ and (81) then imply (36.4.k).

In order to prove (82), pick some small $\zeta_0 = \zeta$ and assume l_0 and a_0 to be so large that for all $\hat{a} \in I_0(a_0)$, $z_i(\hat{a}, l_0) > 1 - \zeta_0$ whenever i is not too near to $l_0 - 1$. Write $\bar{a}$ or $\bar{\bar{a}}$ for any $a \ge a_1$ which satisfies (78). Then

$$z(Z_i(\bar{a})) > 1 - \zeta_0 \tag{83}$$

if i is not too near to $l_0 - 1$. Fix once and for all such an i. Given $\hat{a}_m \in I_m(a_0)$, $m \in \mathbb{N}$, which satisfies (78), set $d = u_{2^m i} - u_{2^m(i-1)}$, $u_M = u_{2^m(i-1)} + d/2$ and $\delta = u_{2^m(i-1)+2^{m-1}} - u_M$, and write $Z_0 = Z_i(\hat{a}_m)$, $Z_1 = \coprod_{j=2^m(i-1)+1}^{2^m(i-1)+2^{m-1}} Z_j(\hat{a}_m, l_m)$ and $Z_2 = Z_0 \setminus Z_1$. Also write $X_j = X(Z_j)$, $Y_j = Y(Z_j)$ and $P_j = P(Z_j)$, $j = 0,1,2$ (see Figure 29). Finally, define $R_h(Z_j)$, $1 \le h \le 4$, in the obvious way.

Define α, β and γ_1 by

$$\frac{|X_1|}{|X_2|} = (1+\alpha)\frac{d/2+\delta}{d/2-\delta}, \tag{84}$$

$$|X_j| \le (1+\beta)|Z_j|, \qquad j = 1,2, \tag{85}$$

and

$$|P_1| = (1+\gamma_1)|P_2|. \tag{86}$$

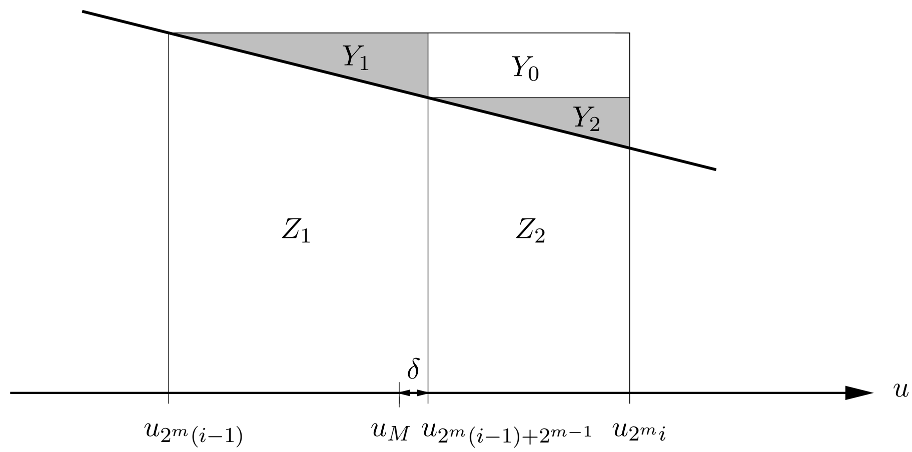

Figure 29: X_0

We assume that β is chosen minimal, and we observe that γ_1 is independent of $\hat{a}_m$ and is small since i is not too near to $l_0 - 1$ and l_0 is large. By (83), $|Z_0| > (1-\zeta_0)|P_0|$. This and (86) readily imply that

$$|Z_j| > (1-(2+\gamma_1)\zeta_0)|P_j|, \qquad j = 1, 2. \tag{87}$$

Thus, since $\gamma_1 < 1$,

$$\begin{aligned}(1+\alpha)\frac{d/2+\delta}{d/2-\delta} &\overset{(84)}{=} \frac{|X_1|}{|X_2|} \overset{(85)}{\geq} \frac{|Z_1|}{(1+\beta)|Z_2|} \\ &\overset{(87)}{>} \frac{(1-3\zeta_0)|P_1|}{(1+\beta)|P_2|} > \frac{1-3\zeta_0}{1+\beta}\end{aligned} \tag{88}$$

and

$$\begin{aligned}\frac{d/2+\delta}{d/2-\delta} < \frac{|X_1|}{|X_2|} &\overset{(85)}{\leq} \frac{(1+\beta)|Z_1|}{|Z_2|} \\ &\overset{(87)}{<} \frac{(1+\beta)|P_1|}{(1-3\zeta_0)|P_2|} \overset{(86)}{=} \frac{(1+\beta)(1+\gamma_1)}{1-3\zeta_0}.\end{aligned} \tag{89}$$

If $\delta < 0$, by (88),

$$d(\alpha+\beta+\alpha\beta+3\zeta_0) > |\delta|(4+2\alpha+2\beta+2\alpha\beta-6\zeta_0),$$

and if $\delta \geq 0$, by (89),

$$d(\gamma_1+\beta+\gamma_1\beta+3\zeta_0) > \delta(4+2\gamma_1+2\beta+2\gamma_1\beta-6\zeta_0).$$

Set $\mu = \max(\alpha, \gamma_1)$. Then

$$|\delta| < \frac{d}{2}(\mu + \beta + 3\zeta_0) \tag{90}$$

if ζ_0, β and μ are small enough.

Set $c = \hat{a}_m - u_{2^m(i-1)}$. Observe that, by (83), if ζ_0 is small, $d(\bar{a})/d(\bar{\bar{a}})$ and $c(\bar{a})/c(\bar{\bar{a}})$ are near to $\bar{a}/\bar{\bar{a}}$ for all $\bar{a}$, $\bar{\bar{a}}$. Hence, $d(\bar{a})/c(\bar{a})$ is essentially independent of $\bar{a}$. Let ν_1 be such that $d(\bar{a})/c(\bar{a}) \leq \nu_1$ for all $\bar{a}$. Since $c(\bar{a})$ is large for i not too near to $l_0 - 1$ and since l_0 is also large, ν_1 is small. Moreover, we readily compute

$$\alpha = \frac{n-1}{2}\frac{d+2\delta}{c} + o\left(\frac{d}{c}\right) \tag{91}$$

and

$$\beta = \frac{n-1}{4}\frac{d+2\delta}{c} + o\left(\frac{d}{c}\right). \tag{92}$$

Thus, α and β are dominated by ν_1, i.e. there are small constants α_1 and β_1 such that $\alpha \leq \alpha_1$ and $\beta \leq \beta_1$ for all $\bar{a}$. Set $\mu_1 = \max(\alpha_1, \gamma_1)$.

Next, notice that $|Y_1|/|P_1|$ and $|Y_2|/|P_2|$ are essentially half as large as $|Y_0|/|P_0|$ and hence also about half as large as $|Y_i(\hat{a}, l_0)|/|P_i(\hat{a}, l_0)|$. Indeed,

$$|Y_0| = \frac{1}{(n-1)!}\left(\frac{\pi c}{\hat{a}_m}\right)^{n-1}\left[d - \frac{c}{n}\left(1 - \left(1 - \frac{d}{c}\right)^n\right)\right],$$

and $|Y_1|$ respectively $|Y_2|$ are obtained from this expression by replacing d by $d/2 + \delta$ respectively c by $c - (d/2 + \delta)$ and d by $d/2 - \delta$. This yields

$$\begin{aligned}\left|\frac{|Y_1|}{|Y_0|} - \frac{1}{4}\right| &= \frac{1}{4}\left|\frac{n-2}{6}\frac{d}{c} + 4\frac{\delta}{d}\right| + o\left(\frac{d}{c}\right) + o\left(\frac{\delta}{d}\right) \\ &\overset{(90)}{<} \frac{n}{2}\nu_1 + \mu_1 + \beta_1 + 3\zeta_0,\end{aligned} \tag{93}$$

and since ν_1 is small, it turns out that the same estimate also holds true for $|Y_2|/|Y_0|$. Moreover, (86) implies that

$$\frac{|P_j|}{|P_0|} \geq \frac{1}{2+\gamma_1}, \qquad j = 1, 2. \tag{94}$$

If ζ_0, β_1, μ_1 and ν_1 and also ϵ are small enough, we hence get

$$\begin{aligned}\frac{|Y_j|}{|P_j|} \overset{(93),(94)}{<} \frac{3}{5}\frac{|Y_0|}{|P_0|} &\overset{(79)}{<} \frac{3}{5}(1+\epsilon_1)\frac{|Y_i(\hat{a},l_0)|}{|Z_i(\hat{a},l_0)|} \\ &< \frac{3}{5}\frac{1+\epsilon_1}{1-\zeta_0}\frac{|Y_i(\hat{a},l_0)|}{|P_i(\hat{a},l_0)|} \\ &< \frac{2}{3}\frac{|Y_i(\hat{a},l_0)|}{|P_i(\hat{a},l_0)|}, \qquad j=1,2.\end{aligned} \tag{95}$$

We conclude that for $j=1,2$

$$\begin{aligned} z(Z_j) = \frac{|Z_j|}{|P_j|} &= 1 - \frac{\sum_{h=1}^4 |R_h(Z_j)|}{|P_j|} \\ &> 1 - \frac{\sum_{h=1}^3 |R_h(Z_j)| + |Y_j|}{|P_j|} \\ &\overset{(77),(95)}{>} 1 - \frac{2}{3}\frac{\sum_{h=1}^4 |R_{h,i}(\hat{a},l_0)|}{|P_i(\hat{a},l_0)|} \\ &> 1 - \frac{2}{3}\zeta_0. \end{aligned} \tag{96}$$

In particular, ζ_0 in (83) may be replaced by $\zeta_1 = \frac{2}{3}\zeta_0$.

We conclude that l_0 and a_0 may be chosen such that for all $\hat{a}_m$

$$(1-L_1)|Y_0| < 4|Y_j| < (1+L_1)|Y_0|, \qquad j=1,2. \tag{97}$$

Here, we put

$$L_1 = L(\zeta_1,\beta_1,\mu_1,\nu_1) = 4(\mu_1+\beta_1+3\zeta_1) + 2n\nu_1.$$

Observe that L is linear in ζ_1, β_1, μ_1 and ν_1.

Assume now that $m \geq 2$ and consider the partition $Z_1 = Z_1^2 \coprod Z_2^2$ whose components consist of 2^{m-2} consecutive components of $Z(\hat{a}_m, l_m)$. Set $d' = d/2 + \delta$ and define δ' to be the difference of the u-width of Z_1^2 and $d'/2$. If α' is defined by

$$\frac{|X_1^2|}{|X_2^2|} = (1+\alpha')\frac{d'/2+\delta'}{d'/2-\delta'},$$

we have

$$\alpha' = \frac{n-1}{2}\frac{d'+2\delta'}{c} + o\left(\frac{d'}{c}\right). \tag{98}$$

Since ζ_1 is small, δ'/d' is small. (91) and (98) thus show that α is near to $2\alpha'$. In particular,

$$\alpha' < \frac{2}{3}\,\alpha. \tag{99}$$

Similarly, if β' is the minimal constant with

$$|X_j^2| \le (1+\beta')|Z_j^2|, \qquad j = 1, 2,$$

we have

$$\begin{aligned}\beta' &= \frac{n-1}{4}\max\left(\frac{d'+2\delta'}{c}, \frac{d'-2\delta'}{c-d'/2-\delta'}\right) + o\left(\frac{d'}{c}\right)\\ &= \frac{n-1}{4}\frac{d'+2|\delta'|}{c} + o\left(\frac{d'}{c}\right),\end{aligned} \tag{100}$$

and we conclude from (92) and (100) as above that

$$\beta' < \frac{2}{3}\,\beta. \tag{101}$$

A similar but simpler calculation shows that γ', which is defined by $P(Z_1^2) = (1+\gamma')P(Z_2^2)$, satisfies

$$\gamma' < \frac{2}{3}\,\gamma_1. \tag{102}$$

Next, since δ'/d is small, we also have that

$$\frac{d'}{c} < \frac{2}{3}\,\nu_1. \tag{103}$$

Consider now the partition $Z_2 = Z_3^2 \coprod Z_4^2$. While for Z_1 we had $c' = c$, now, $c'' = \hat{a}_m - u_{2^m(i-1)+2^{m-1}} = c - d'$. But $c''/c = 1 - d'/c$ is near to 1, whence the same arguments as above show (99), (101), (102) and (103) with α', β', γ' and c' replaced by α'', β'', γ'' and c''. Finally, an argument analogous to the one which proved (96) shows $z(Z_j^2) > 1 - \frac{2}{3}\zeta_1$, $1 \le j \le 4$. Summing up, we have shown that there are constants $\zeta_2 = \frac{2}{3}\zeta_1$, β_2, μ_2 and ν_2 independent of $\hat{a}_m$ such that $L_2 = L(\zeta_2, \beta_2, \mu_2, \nu_2)$ satisfies $L_2 < \frac{2}{3}L_1$ and such that for all $\hat{a}_m$

$$(1-L_2)|Y_j| < 4|Y^2_{2j(-1)}| < (1+L_2)|Y_j|, \qquad j = 1, 2.$$

In general, let $Z^k(\hat{a}_m)$, $0 \le k \le m$, be the partition of Z_0 whose components consist of 2^{m-k} consecutive components of $Z(\hat{a}_m, l_m)$. Applying the

above arguments to the components of $Z^k(\hat{a}_m)$, we see by finite induction that there are constants L_k, $1 \le k \le m$, with $L_{k+1} < \frac{2}{3} L_k$ such that for all $\hat{a}_m$

$$(1 - L_{k+1})|Y_j^k| < 4|Y_{2j(-1)}^{k+1}| < (1 + L_{k+1})|Y_j^k)|,$$

$1 \le j \le 2^k$, $0 \le k \le m-1$. Hence, with

$$\pi_\pm(x) = \prod_{k=1}^{\infty} \left(1 \pm \left(\frac{2}{3}\right)^k x\right)$$

we have that for all $j \in \{1, \dots, 2^m\}$

$$\begin{aligned} \pi_-(L_1)|Y_0| \; &< \; \prod_{k=1}^{m} (1 - L_k)|Y_0| \\ &< \; 4^m |Y_j^m| \\ &< \; \prod_{k=1}^{m} (1 + L_k)|Y_0| \; < \; \pi_+(L_1)|Y_0|. \end{aligned} \tag{104}$$

Let l_0 and a_0 be so large that for i not too near to $l_0 - 1$, L_1 is so small that $1 - \epsilon_2 < \pi_-(L_1)$ and $\pi_+(L_1) < 1 + \epsilon_2$. Then (104) implies (82). This completes the proof of Proposition 3.14. $\square$

3.4 Lagrangian folding

As already mentioned at the beginning of this section, there is a Lagrangian version of folding developed by Traynor in [31]. Here, the whole ellipsoid or the whole polydisc is viewed as a Lagrangian product of a cube and a simplex or a cube, and folding is then simply achieved by wrapping the base cube around the base of the cotangent bundle of the torus via a linear map. This version has thus a more algebraic flavour. However, it yields good embeddings only for comparable shapes, while the best embeddings of an ellipsoid into a polydisc respectively of a polydisc into an ellipsoid via Lagrangian folding pack less than $1/n!$ respectively $n!/n^n$ of the volume.

For the convenience of the reader we review the method briefly.
Write again $\mathbb{R}^{2n}(x, y) = \mathbb{R}^n(x) \times \mathbb{R}^n(y)$ and set

$$\square(a_1, \dots, a_n) = \{0 < x_i < a_i, \; 1 \le i \le n\} \subset \mathbb{R}^n(x),$$

$$\triangle(b_1, \dots, b_n) = \left\{ 0 < y_1, \dots, y_n \;\middle|\; \sum_{i=1}^{n} \frac{y_i}{b_i} < 1 \right\} \subset \mathbb{R}^n(y)$$

and

$$T^n = \mathbb{R}^n(x)/\pi\mathbb{Z}^n.$$

The embeddings are given by the compositions of maps

$$\begin{array}{rcl} E(a_1-\epsilon,\dots,a_n-\epsilon) & \xrightarrow{\alpha_E} & \square^n(1)\times\triangle(a_1,\dots,a_n) \\ & \xrightarrow{\beta} & \square(q_1\pi,\dots,q_n\pi)\times\triangle(\frac{a_1}{q_1\pi},\dots,\frac{a_n}{q_n\pi}) \\ & \xrightarrow{\gamma} & T^n\times\triangle^n(\frac{A}{\pi}) \\ & \xrightarrow{\delta_E} & B^{2n}(A) \end{array}$$

respectively

$$\begin{array}{rcl} P(a_1,\dots,a_n) & \xrightarrow{\alpha_P} & \square^n(1)\times\square(a_1,\dots,a_n) \\ & \xrightarrow{\beta} & \square(q_1\pi,\dots,q_n\pi)\times\square(\frac{a_1}{q_1\pi},\dots,\frac{a_n}{q_n\pi}) \\ & \xrightarrow{\gamma} & T^n\times\square^n(\frac{A}{\pi}) \\ & \xrightarrow{\delta_P} & C^{2n}(A), \end{array}$$

where $\epsilon > 0$ is arbitrarily small and the q_i are of the form k_i or $1/k_i$ for some $k_i \in \mathbb{N}$.

α_E and α_P are the map $(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n)$ followed by the maps described at the beginning of section 3.3, and β is a diagonal linear map:

$$\beta = \mathrm{diag}\left[q_1\pi,\dots,q_n\pi,\frac{1}{q_1\pi},\dots,\frac{1}{q_n\pi}\right].$$

Next, let

$$\tilde{\delta}_E \colon \square^n(\pi)\times\triangle^n\left(\frac{A}{\pi}\right) \hookrightarrow B^{2n}(A)$$

and

$$\tilde{\delta}_P \colon \square^n(\pi)\times\square^n\left(\frac{A}{\pi}\right) \hookrightarrow C^{2n}(A)$$

be given by

$$\begin{array}{rcl} (x_1,\dots,x_n,y_1,\dots,y_n) & \mapsto & (\sqrt{y_1}\cos 2x_1,\dots,\sqrt{y_n}\cos 2x_n, \\ & & -\sqrt{y_1}\sin 2x_1,\dots,-\sqrt{y_n}\sin 2x_n). \end{array}$$

Notice that $\tilde{\delta}_E$ respectively $\tilde{\delta}_P$ extend to an embedding of $T^n \times \triangle^n(A/\pi)$ respectively $T^n \times \square^n(A/\pi)$. These extensions are the maps δ_E and δ_P. We finally come to the folding map γ.

Lemma 3.15 *(i) If the natural numbers $k_1, \dots, k_{n-1}$ are relatively prime, then*

$$M(k_1, \dots, k_{n-1}) = \begin{pmatrix} 1 & & & -\frac{1}{k_1} \\ & 1 & 0 & -\frac{1}{k_2} \\ & & \ddots & \vdots \\ & 0 & 1 & -\frac{1}{k_{n-1}} \\ & & & 1 \end{pmatrix}$$

embeds $\square(\pi/k_1, \dots, \pi/k_{n-1}, k_1 \dots k_{n-1}\pi)$ into T^n.

(ii) For any $k_2, \dots, k_n \in \mathbb{N} \setminus \{1\}$

$$N(k_2, \dots, k_n) = \begin{pmatrix} 1 & -\frac{1}{k_2} & & & \\ & 1 & -\frac{1}{k_3} & 0 & \\ & & \ddots & \ddots & \\ & & & \ddots & -\frac{1}{k_{n-1}} & \\ & 0 & & & 1 & -\frac{1}{k_n} \\ & & & & & 1 \end{pmatrix}$$

embeds $\square(\pi/(k_2 \dots k_n), k_2\pi, \dots, k_n\pi)$ into T^n.

Proof. ad (i). Let $Mx = Mx'$ for $x, x' \in \square(1/k_1, \dots, 1/k_{n-1}, k_1 \dots k_{n-1})$, so

$$x_i - \frac{x_n}{k_i} = x_i' - \frac{x_n'}{k_i} + l_i, \qquad 1 \le i \le n-1 \tag{105}$$

for some $l_i \in \mathbb{Z}$ and

$$x_n = x_n' + l_n, \tag{106}$$

where $l_n \in \mathbb{Z}$ satisfies $|l_n| < k_1 \dots k_{n-1}$. Substituting (106) into (105) we get

$$x_i - x_i' = l_i + \frac{l_n}{k_i}, \qquad 1 \le i \le n-1. \tag{107}$$

If $l_n = 0$, we conclude $x = x'$. Otherwise, $|x_i - x_i'| < 1/k_i$ for $1 \le i \le n-1$ and (107) imply that l_n is an integral multiple of all the k_i, whence by the assumption on the k_i we have $|l_n| \ge k_1 \dots k_{n-1}$, a contradiction.

ad (ii). Let $Nx = Nx'$ for $x, x' \in \square(1/(k_2 \dots k_n), k_2, \dots, k_n)$, so

$$x_i - \frac{x_{i+1}}{k_{i+1}} = x_i' - \frac{x_{i+1}'}{k_{i+1}} + l_i, \qquad 1 \le i \le n-1 \tag{108}$$

for some $l_i \in \mathbb{Z}$ and

$$x_n = x_n' + l_n. \tag{109}$$

Substituting (109) into the last equation of (108) and resubstituting the resulting equations successively into the preceding ones, we get

$$x_1 = x_1' + \frac{l_n}{k_2 \dots k_n} + \frac{l_{n-1}}{k_2 \dots k_{n-1}} + \frac{l_{n-2}}{k_2 \dots k_{n-2}} + \dots + \frac{l_2}{k_2} + l_1. \tag{110}$$

Since $|x_1 - x_1'| < 1/(k_2 \dots k_n)$, equation (110) has no solution for $x_1 \neq x_1'$, hence $x_1 = x_1'$, and substituting this into (108) and using $|x_i - x_i'| < k_i$, $2 \le i \le n$, we successively find $x_i = x_i'$. $\square$

The folding map γ can thus be taken to be $M \times M^*$, where M is as in (i) or (ii) of the lemma and M^* denotes the transpose of the inverse of M.

Remark 3.16 For polydiscs, the construction clearly commutes with taking products. For ellipsoids, a similar compatibility holds: Let M_1^* respectively M_2^* be linear injections of $\triangle(a_1, \dots, a_m)$ into $\triangle(a_1', \dots, a_m')$ respectively $\triangle(b_1, \dots, b_n)$ into $\triangle(b_1', \dots, b_n')$. Then $M_1^* \oplus M_2^*$ clearly injects $\triangle(a_1, \dots, a_m, b_1, \dots, b_n)$ into $\triangle(a_1', \dots, a_m', b_1', \dots, b_n')$. Thus, given (possibly trivial) Lagrangian foldings λ_1 and λ_2 which embed $E(a_1, \dots, a_m)$ into $E(a_1', \dots, a_m')$ and $E(b_1, \dots, b_n)$ into $E(b_1', \dots, b_n')$, the Lagrangian folding $\lambda_1 \oplus \lambda_2$ embeds $E(a_1, \dots, a_m, b_1, \dots, b_n)$ into $E(a_1', \dots, a_m', b_1', \dots, b_n')$. $\diamond$

In the following statements, ϵ denotes any positive number.

Proposition 3.17 *(i) Let $k_1 < \dots < k_{n-1}$ be relatively prime and $a > 0$. Then*

$(i)_E$ $E^{2n}(\pi, \dots, \pi, a) \hookrightarrow B^{2n}(\max\{(k_{n-1}+1)\pi, \frac{a}{k_1 \cdots k_{n-1}}\} + \epsilon)$

$(i)_P$ $P^{2n}(\pi, \dots, \pi, a) \hookrightarrow C^{2n}(\max\{k_{n-1}\pi, (n-1)\pi + \frac{a}{k_1 \cdots k_{n-1}}\})$.

(ii) Let $n \ge 3$, $k_2, \dots, k_n \in \mathbb{N} \setminus \{1\}$ and $a_2, \dots, a_n > 0$. Then

$(ii)_E$ *$E(\pi, a_2, \dots, a_n) \hookrightarrow B^{2n}(A+\epsilon)$, where A is found as follows: Multiply the first column of N^* by $k_2 \cdots k_n$ and the i th column by $(a_i/\pi)/k_i$, $2 \le i \le n$. Then add to every row its smallest entry and add up the entries of each column. A/π is the maximum of these sums.*

$(ii)_P$ *$P(\pi, a_2, \dots, a_n) \hookrightarrow P(A_1, \dots, A_n)$, where the A_i are found as follows: Multiply N^* as in $(ii)_E$. A_i/π is the sum of the absolute values of the entries of the i th row.*

Proof. ad (i). Write $y' = M^*(k_1, \dots, k_{n-1})y$. We have

$$M^*(k_1, \dots, k_{n-1}) = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & & & 1 & \\ \frac{1}{k_1} & \frac{1}{k_2} & \cdots & \frac{1}{k_{n-1}} & 1 \end{pmatrix}.$$

Thus, given $y \in \triangle(k_1, \dots, k_{n-1}, \frac{a/\pi}{k_1 \cdots k_{n-1}})$,

$$\begin{aligned} y_1' + \cdots + y_n' &= (k_1+1)\frac{y_1}{k_1} + \cdots + (k_{n-1}+1)\frac{y_{n-1}}{k_{n-1}} + \frac{a/\pi}{k_1 \cdots k_{n-1}} \frac{y_n}{\frac{a/\pi}{k_1 \cdots k_{n-1}}} \\ &< \max\left\{k_{n-1}+1, \frac{a/\pi}{k_1 \cdots k_{n-1}}\right\}, \end{aligned}$$

and given $y \in \square(k_1, \dots, k_{n-1}, \frac{a/\pi}{k_1 \cdots k_{n-1}})$,

$$y' \in \square(k_1, \dots, k_{n-1}, n-1+\frac{a/\pi}{k_1 \cdots k_{n-1}}).$$

ad (ii). We have

$$N^*(k_2, \dots, k_n) = \begin{pmatrix} 1 & & & & & \\ \frac{1}{k_2} & 1 & & & 0 & \\ -\frac{1}{k_2 k_3} & \frac{1}{k_3} & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \frac{(-1)^{n-1}}{k_2 \cdots k_{n-1}} & \frac{(-1)^{n-2}}{k_3 \cdots k_{n-1}} & \cdots & \frac{1}{k_{n-1}} & 1 & \\ \frac{(-1)^n}{k_2 \cdots k_n} & \frac{(-1)^{n-1}}{k_3 \cdots k_n} & \cdots & \frac{-1}{k_{n-1}k_n} & \frac{1}{k_n} & 1 \end{pmatrix}.$$

Observe that we are free to compose N^* with a translation. Multiplying the columns as prescribed we get the vertices of the simplex

$$N^* \triangle \left(k_2 \dots k_n, \frac{a_2/\pi}{k_2}, \dots, \frac{a_n/\pi}{k_n} \right).$$

Adding to the rows of this new matrix its smallest entry corresponds to translating this new simplex into the positive cone of $\mathbb{R}^n(y)$. The claim thus follows. A similar but simpler procedure leads to the last statement. $\square$

Proposition 3.17 leads to the number theoretic problem of finding appropriate relatively prime numbers $k_1, \dots, k_{n-1}$. An effective method which solves this problem for a large is described in the proof of Proposition 4.10 $(\mathrm{i})_E$.

Corollary 3.18 *$(i)_E$ $E^{2n}(\pi, l_{EB}(a), \dots, l_{EB}(a), a) \hookrightarrow B^{2n}(l_{EB}(a)+\epsilon)$, where*

$$l_{EB}(a) = \min_{k \in \mathbb{N}} \max\{(k+1)\pi, a/k\} = \begin{cases} (k+1)\pi, & (k-1)(k+1) \leq a/\pi \leq k(k+1) \\ a/k, & k(k+1) \leq a/\pi \leq k(k+2). \end{cases}$$

$(i)_P$ $P^{2n}(\pi, l_{PC}(a), \dots, l_{PC}(a), a) \hookrightarrow C^{2n}(l_{PC}(a))$, where

$$l_{PC}(a) = \min_{k \in \mathbb{N}} \max\{k\pi, a/k + \pi\} = \begin{cases} k\pi, & (k-1)^2 \leq a/\pi \leq k(k-1) \\ a/k + \pi, & k(k-1) \leq a/\pi \leq k^2. \end{cases}$$

For $n \geq 3$ and any $k \in \mathbb{N} \setminus \{1\}$

$(ii)_E$ $E^{2n}(\pi, k^n\pi, \dots, k^n\pi) \hookrightarrow B^{2n}((k^{n-1} + k^{n-2} + (n-2)k^{n-3})\pi + \epsilon)$,

$(ii)_P$ $P^{2n}(\pi, (k-1)k^{n-1}\pi, \dots, (k-1)k^{n-1}\pi) \hookrightarrow C^{2n}(k^{n-1}\pi)$.

Proof. In $(\mathrm{i})_E$ and $(\mathrm{i})_P$ Remark 3.16 was applied. For both $(\mathrm{ii})_E$ and $(\mathrm{ii})_P$ choose $k_2 = \cdots = k_n = k$. In $(\mathrm{ii})_E$, the maximal sum is the one of the entries of the $n-1$ st column, and in $(\mathrm{ii})_P$ all the sums are k^{n-1}. $\square$

Examples.

ad $(\mathrm{i})_E$ and $(\mathrm{i})_P$. Remark 3.16 and Proposition 3.17 (i) applied to opposite entries imply that for any $k \in \mathbb{N}$

$$E^{2n}(\pi, k\pi, k^2\pi, \dots, k^{2l}\pi) \hookrightarrow B^{2n}((k^l + k^{l-1})\pi + \epsilon)$$

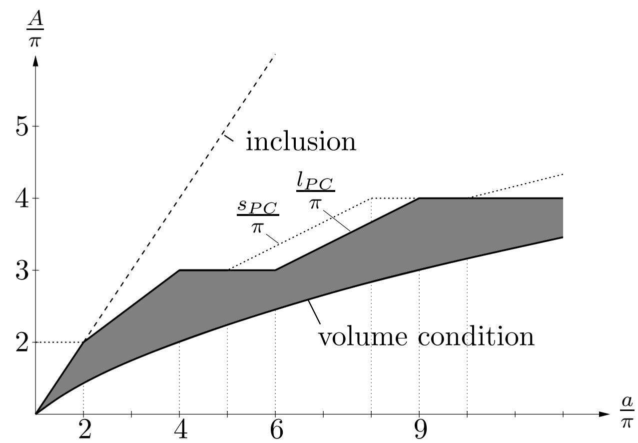

Figure 30: What is known about $P(\pi, a) \hookrightarrow C^4(A)$

and

$$P^{2n}(\pi, k\pi, k^2\pi, \dots, k^{2l}\pi) \hookrightarrow C^{2n}((k^l + k^{l-1})\pi)$$

if $n = 2l + 1$ is odd and

$$E^{2n}(\pi, k^2\pi, k^4\pi, \dots, k^{2n-2}\pi) \hookrightarrow B^{2n}((k^{n-1} + k^{n-2})\pi + \epsilon)$$

and

$$P^{2n}(\pi, k^2\pi, k^4\pi, \dots, k^{2n-2}\pi) \hookrightarrow C^{2n}((k^{n-1} + k^{n-2})\pi)$$

if n is even.

ad (ii)$_E$. For $n = 3$, Proposition 3.17 yields

$$E(\pi, a_2, a_3) \hookrightarrow B^6\left(\max\left\{k_3(k_2+1)\pi, \frac{a_2}{k_2 k_3}(k_3+1) + \pi, \frac{a_3}{k_3} + \pi\right\} + \epsilon\right)$$

for any $k_2, k_3 \in \mathbb{N} \setminus \{1\}$. With $(k_2, k_3) = (k, lk-1)$ we thus get for any $k \in \mathbb{N} \setminus \{1\}$ and $l \in \mathbb{N}$

$$E\left(\pi, \frac{k(lk-1)^2}{l}\pi, k(lk-1)^2\pi\right) \hookrightarrow B^6(k(lk-1)\pi + \pi + \epsilon).$$

ad (ii)$_P$. For $n = 3$, Proposition 3.17 yields

$$P(\pi, a_2, a_3) \hookrightarrow C^6\left(\max\left\{k_2 k_3\pi, k_3\pi + \frac{a_2}{k_2}, \pi + \frac{a_2}{k_2 k_3} + \frac{a_3}{k_3}\right\}\right)$$

for any $k_2, k_3 \in \mathbb{N} \setminus \{1\}$. With $(k_2, k_3) = (k, lk - l + 1)$ we thus get for any $k \in \mathbb{N} \setminus \{1\}$ and $l \in \mathbb{N}$

$$P(\pi, (k-1)k(lk-l+1)\pi, l(k-1)k(lk-l+1)\pi) \hookrightarrow C^6(k(lk-l+1)\pi).$$

$\diamond$

3.5 Symplectic versus Lagrangian folding

For small a, the estimate s_{EB} provides the best result known. For example, we get $\frac{s_{EB}}{\pi}(4\pi) = 2.6916\ldots$, whence we have proved

Fact. *$E(\pi, 4\pi)$ embeds in $B^4(2.692\,\pi)$.*

$l_{EB}(a) < s_{EB}(a)$ happens first at $a/\pi = 5.1622\ldots$. In general, computer calculations suggest that l_{EB} and s_{EB} yield alternately better estimates: For all $k \in \mathbb{N}$ we seem to have that $l_{EB} < s_{EB}$ on an interval around $a = k(k+1)\pi$ and $s_{EB} < l_{EB}$ on an interval around $k(k+2)\pi$; moreover, they suggest that

$$\lim_{k \to \infty} (s_{EB}(k(k+2)\pi) - l_{EB}(k(k+2)\pi)) = 0,$$

i.e. l_{EB} and s_{EB} seem to be asymptotically equivalent. We checked the above statements for $k \leq 5\,000$.

Remark 3.19 The difference $d_{EB}(a) = l_{EB}(a) - \sqrt{\pi a}$ between l_{EB} and the volume condition attains local maxima at $a_k = k(k+2)\pi$, where $d_{EB}(a) = (k+2)\pi - \sqrt{k(k+2)}\,\pi$. This is a decreasing sequence converging to π. $\diamond$

Figure 1 summarizes the results. The non trivial estimates from below are provided by Ekeland-Hofer capacities, which yield $A(a) \geq a$ for $a \in [\pi, 2\pi]$ and $A(a) \geq 2\pi$ for $a > 2\pi$.

3.6 Summary

Given $U \in \mathcal{O}(n)$ and $\alpha > 0$, set $\alpha U = \{\alpha z \in \mathbb{C}^n \mid z \in U\}$.

For $U, V \in \mathcal{O}(n)$ define squeezing constants

$$s(U,V) = \inf\{\alpha \mid \text{there is a symplectic embedding } \varphi\colon U \hookrightarrow \alpha V\}.$$

Specializing, we define *squeezing numbers*

$$s^E_{q_2\dots q_n}(U) = s(U, E(1, q_2, \dots, q_n))$$

and

$$s^P_{q_2\dots q_n}(U) = s(U, P(1, q_2, \dots, q_n)),$$

and we write $s^B(U)$ for $s^E_{1\dots 1}(U)$ and $s^C(U)$ for $s^P_{1\dots 1}(U)$.

With this notation, the main results of this section read

$$s^B(E(\pi, a)) \leq \min(s_{EB}(a), l_{EB}(a)) \tag{111}$$
$$s^B(P(\pi, a)) \leq s_{PB}(a) \tag{112}$$
$$s^C(E(\pi, a)) \leq s_{EC}(a) \tag{113}$$
$$s^C(P(\pi, a)) \leq \min(s_{PC}(a), l_{PC}(a)) \tag{114}$$

and

$$s^C(P^{2n}(\pi, \dots, \pi, a)) \leq s^{2n}_{PC}(a)$$

4 Packings

In the previous section we tried to squeeze a given simple shape into a minimal ball and a minimal cube. This problem may be reformulated as follows:

"Given a ball B respectively a cube C and a simple shape S, what is the largest simple shape similar to S which fits into B respectively C?"

or equivalently:

"Given a ball or a cube, how much of its volume may be symplectically packed by a simple shape of a given shape?"

More generally, given $U \in \mathcal{O}(n)$ and any connected symplectic manifold (M^{2n}, ω), define the *U-width* of (M, ω) by

$$w(U, (M, \omega)) = \sup\{\alpha \mid \text{there is a symplectic embedding } \varphi\colon \alpha U \hookrightarrow (M, \omega)\},$$

and if the volume $\mathrm{Vol}(M, \omega) = \frac{1}{n!}\int_M \omega^n$ is finite, set

$$p(U, (M, \omega)) = \frac{|w(U, (M, \omega))U|}{\mathrm{Vol}(M, \omega)}.$$

In this case, the two invariants determine each other, $p(U,(M,\omega)) > 0$ by Darboux's theorem, and if in addition $n = 1$, $p(U,(M,\omega)) = 1$ by Theorem 4.2.

Given real numbers $1 \le q_2 \le \cdots \le q_n$, we define *weighted widths*

$$\begin{aligned} w^E_{q_2\dots q_n}(M,\omega) &= w(E(1,q_2,\dots,q_n),(M,\omega)), \\ w^P_{q_2\dots q_n}(M,\omega) &= w(P(1,q_2,\dots,q_n),(M,\omega)) \end{aligned}$$

and *packing numbers*

$$\begin{aligned} p^E_{q_2\dots q_n}(M,\omega) &= p(E(1,q_2,\dots,q_n),(M,\omega)) = \frac{(w^E_{q_2\dots q_n}(M,\omega))^n q_2\dots q_n}{n!\,\mathrm{Vol}(M,\omega)}, \\ p^P_{q_2\dots q_n}(M,\omega) &= p(P(1,q_2,\dots,q_n),(M,\omega)) = \frac{(w^P_{q_2\dots q_n}(M,\omega))^n q_2\dots q_n}{\mathrm{Vol}(M,\omega)}. \end{aligned}$$

Write $w(M,\omega)$ for the Gromov width $w^E_{1\dots 1}(M,\omega)$ and $p(M,\omega)$ for $p^E_{1\dots 1}(M,\omega)$.

Example 4.1 Assume that $(M,\omega) = (V,\omega_0) \in \mathcal{O}(n)$. By the very definitions of squeezing constants and widths we have

$$w(U,V) = \frac{1}{s(U,V)}.$$

In particular, we see that squeezing numbers and weighted widths of simple shapes determine each other via

$$w^E_{q_2\dots q_n}(E(\pi,p_2\pi,\dots,p_n\pi)) = \frac{\pi^2}{s^E_{p_2\dots p_n}(E(\pi,q_2\pi,\dots,q_n\pi))}, \quad (115)$$

$$w^P_{q_2\dots q_n}(P(\pi,p_2\pi,\dots,p_n\pi)) = \frac{\pi^2}{s^P_{p_2\dots p_n}(P(\pi,q_2\pi,\dots,q_n\pi))}, \quad (116)$$

$$w^E_{q_2\dots q_n}(P(\pi,p_2\pi,\dots,p_n\pi)) = \frac{\pi^2}{s^P_{p_2\dots p_n}(E(\pi,q_2\pi,\dots,q_n\pi))}, \quad (117)$$

$$w^P_{q_2\dots q_n}(E(\pi,p_2\pi,\dots,p_n\pi)) = \frac{\pi^2}{s^E_{p_2\dots p_n}(P(\pi,q_2\pi,\dots,q_n\pi))}. \quad (118)$$

Combined with the estimates stated in subsection 3.6, these equations provide estimates of weighted widths and packing numbers of simple shapes from below. ◇

If (M,ω) is an arbitrary symplectic manifold whose Gromov width is known to be large, these results may be used to estimate $w^E_{q_2 \dots q_n}(M,\omega)$ and $p^E_{q_2 \dots q_n}(M,\omega)$ reasonably well from below.

Example. Let $T^2(\pi)$ be the 2-torus of volume π and $S^2(2\pi)$ the sphere of volume 2π and endow $M = T^2(\pi) \times S^2(2\pi)$ with the split symplectic structure. Theorem 5.2(ii) shows that $p(M) = 1$. Thus, by (115) and (111)

$$\begin{aligned} p^E_q(M) \geq p^E_q(B^4(2\pi)) &= \frac{(w^E_q(B^4(2\pi)))^2 q}{4\pi^2} \\ &= \frac{q\pi^2}{(s^B(E(\pi,q\pi)))^2} \geq \frac{q\pi^2}{(\min(s_{EB}(q\pi), l_{EB}(q\pi)))^2}. \end{aligned}$$

In particular, $\lim_{q\to\infty} p^E_q(M) = 1$. $\diamond$

On the other hand, $w(U,(M,\omega)) \geq w(V,(M,\omega))$ whenever $U \leq_3 V$; in particular, $w \geq w^E_{q_2 \dots q_n} \geq w^P_{q_2 \dots q_n}$ for all $1 \leq q_2 \leq \dots \leq q_n$. Thus, if $w(M,\omega)$ and the weights are small, we get good estimates of weighted widths and packing numbers from above.

Example. Let $r \geq 1$ and $M = S^2(\pi) \times S^2(r\pi)$ with the split symplectic structure. By the Non-Squeezing Theorem stated at the beginning of Appendix B we have $w(M) \leq \pi$, whence $w^E_q(M) \leq \pi$ and $p^E_q(M) \leq \frac{q}{2r}$. For $q \leq r$ the obvious embedding $E(\pi,q\pi) \hookrightarrow P(\pi,r\pi) \hookrightarrow M$ shows that these inequalities are actually equalities. $\diamond$

The knowledge of the Gromov width is thus of particular importance to us. Recently considerable progress has been made in computing or estimating the Gromov width of closed 4-manifolds. An overview on these results is given in Appendix B.

Remark. Since the Gromov width is the smallest symplectic capacity we might try to estimate it from above by using other symplectic capacities. However, other capacities (like the Hofer-Zehnder capacity or the first Ekeland-Hofer capacity, Viterbo's capacity and the capacity arising from symplectic homology in the case of subsets of $\mathbb{R}^{2n}$) are usually even harder to compute. In fact, we do not know of any space for which a capacity other than the Gromov width is known and finite while its Gromov width

is unknown. $\diamond$

4.1 Asymptotic packings

Theorem 4.2 *Let M^n be a connected manifold endowed with a volume form Ω and let $U \subset \mathbb{R}^n$ be diffeomorphic to a standard ball. Then U embeds in M by a volume preserving map if and only if $|U| \leq \mathit{Vol}(M, \Omega)$.*

Proof. Endow $\overline{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{\infty\}$ with the topology whose base of open sets is given by joining the open intervals $]a, b[\subset \mathbb{R}_{>0}$ with the subsets of the form $]a, \infty] =]a, \infty[\cup \{\infty\}$. Denote the Euclidean norm on $\mathbb{R}^n$ by $\|\cdot\|$ and let S_1 be the unit sphere in $\mathbb{R}^n$.

Lemma 4.3 *Let $\mathbb{R}^n$ be endowed with its standard smooth structure, let $\mu\colon S_1 \to \overline{\mathbb{R}}_{>0}$ be a continuous function and let*

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; x = 0 \;\text{ or }\; 0 < \|x\| < \mu\left(\frac{x}{\|x\|}\right) \right\}$$

be the starlike domain associated to μ. Then S is diffeomorphic to $\mathbb{R}^n$.

Remark. The diffeomorphism guaranteed by the lemma may be chosen such that the rays emanating from the origin are preserved.

Proof of the lemma. If $\mu(S_1) = \{\infty\}$, there is nothing to prove. For μ bounded, the lemma was proved by Ozols [28]. If μ is neither bounded nor $\mu(S_1) = \{\infty\}$, Ozols's proof readily extends to our situation. Using his notation, the only modifications needed are: Require in addition that $r_0 < 1$ and that $\epsilon_1 < 2$, and define continuous functions $\tilde{\mu}_i\colon S_1 \to \mathbb{R}_{>0}$ by

$$\tilde{\mu}_i = \min\{i, \mu - \epsilon_i + \delta_i/2\}.$$

With these minor adaptations the proof in [28] applies word by word. $\square$

Next, pick a complete Riemannian metric g on M. (We refer to [16] for basic notions and results in Riemannian geometry.) The existence of such a metric is guaranteed by a theorem of Whitney [33], according to which M can be embedded as a closed submanifold in some $\mathbb{R}^m$. We may thus take the induced Riemannian metric. A direct and elementary proof of the existence of a complete Riemannian metric is given in [27]. Fix a point $p \in M$, let $\exp_p\colon T_pM \to M$ be the exponential map at p with respect to g,

let $C(p)$ be the cut locus at p and set $\widetilde{C}(p) = \exp_p^{-1}(C(p))$. Let S_1 be the unit sphere in T_pM, let $\mu_p \colon S_1 \to \overline{\mathbb{R}}_{>0}$ be the function defining $\widetilde{C}(p)$ and let $S_p \subset T_pM$ be the starlike domain defined by $\widetilde{C}(p)$. Since g is complete, μ_p is continuous [16, p. 98]. We are thus in the situation of Lemma 4.3, and since $\exp_p(S_p) = M \setminus C(p)$ [16, p. 100], we obtain

Corollary 4.4 *Let (M^n, g) be a complete Riemannian manifold. Then the maximal normal neighbourhood $M \setminus C(p)$ of any point p in M is diffeomorphic to the standard $\mathbb{R}^n$.*

Using polar coordinates on T_pM we see from Fubini's Theorem that $\widetilde{C}(p)$ has zero measure; thus the same holds true for $C(p)$, whence

$$\mathrm{Vol}\,(S_p, \exp_p^* \Omega) = \mathrm{Vol}\,(M \setminus C(p), \Omega) = \mathrm{Vol}\,(M, \Omega).$$

Theorem 4.2 now follows from Lemma 4.3 and

Proposition 4.5 (Greene-Shiohama, [11]) *Two volume forms Ω_1 and Ω_2 on an open manifold are diffeomorphic if and only if the total volume and the set of ends of infinite volume are the same for both forms.*

$\square$

Remark. The existence of a volume preserving embedding of a set U as above with $|U| < \mathrm{Vol}\,(M, \Omega)$ immediately follows from Moser's deformation technique if M is closed and from Proposition 4.5, which is itself an extension of that technique to open manifolds, if M is open. The main point in Theorem 4.2, however, is that *all* of the volume of M can be filled. This is in contrast to the full symplectic packings by k balls established in [25], [2] and [3]. $\diamond$

In view of the Non-Squeezing Theorem and the existence of symplectic capacities, very much in contrast to the volume-preserving case, there exist strong obstructions to full packings by "round" simple shapes in the symplectic category. (We refer to the previous sections for related results on embeddings into simple shapes and to Appendix B for an overview on known results on the Gromov width of closed four manifolds.)

However, the results of section 3 show for example that for embeddings into four dimensional simple shapes packing obstructions more and more disappear if we pass to skinny domains. The main goal of this section is to show that in the limit rigidity indeed disappears.

Theorem 4.6 *Let (M,ω) be a connected symplectic manifold of finite volume. Then*

$$p_\infty^E(M,\omega) = \lim_{q\to\infty} p^E_{1\dots1q}(M,\omega) \quad \textit{and} \quad p_\infty^P(M,\omega) = \lim_{q\to\infty} p^P_{1\dots1q}(M,\omega)$$

exist and equal 1.

Remark. Remark 3.16, Proposition 3.17(i) and the theorem immediately imply that for any (M,ω) as in the theorem

$$\lim_{q\to\infty} p^E_{qq^2\dots q^{n-1}}(M,\omega) \quad \text{and} \quad \lim_{q\to\infty} p^P_{qq^2\dots q^{n-1}}(M,\omega)$$

exist and equal 1. $\diamond$

The proof of the statement for polydiscs proceeds along the following lines: We first fill M up to some ϵ with small disjoint closed cubes, which we connect by lines. We already know how to asymptotically fill these cubes with thin polydiscs, and we may use neighbourhoods of the lines to pass from one cube to another (cf. Figure 31).

The case of ellipsoids is less elementary. For $n \le 3$, the statement for ellipsoids follows from the one for polydiscs and the fact that a polydisc may be asymptotically filled by skinny ellipsoids. This is proved in the same way as (26). In higher dimensions, however, symplectic folding alone is not powerful enough to fill a polydisc by thin ellipsoids, since there is no elementary way of filling a cube by balls. However, algebro-geometric methods imply that in any dimension cubes can indeed be filled by balls. Using this, we may almost fill (M,ω) by equal balls, which we connect again by thin lines. The claim then readily follows from the proof of Proposition 3.13.

We begin with the following

Lemma 4.7 (McDuff-Polterovich, [25]) *Let (M,Ω) be a symplectic manifold of finite volume. Then, given $\epsilon > 0$, there is an embedding of a disjoint union of closed equal cubes $\coprod \overline{C}(\lambda)$ into M such that $|\coprod C(\lambda)| >$ Vol $(M) - 2\epsilon$.*

Proof. Assume first that M is compact and cover M with Darboux charts $V_i = \varphi_i(U_i)$, $i = 1,\dots,m$. Pick closed cubes $\overline{C}_1,\dots,\overline{C}_{j_1} \subset U_1$ of possibly varying size such that

$$\sum_{j=1}^{j_1} |C_j| > \mathrm{Vol}\,(V_1) - \frac{\epsilon}{m}.$$

Proceeding by finite induction, for $i > 1$, set $k_i = \sum_{l=1}^{i-1} j_l$ and pick closed cubes $\overline{C}_{k_i+1}, \dots, \overline{C}_{k_i+j_i} \subset U_i \setminus \varphi_i^{-1}(\bigcup_{j=1}^{i-1} V_j)$ such that

$$\sum_{j=1}^{j_i} |C_{k_i+j}| > \operatorname{Vol}(V_i \setminus \bigcup_{j=1}^{i-1} V_j) - \frac{\epsilon}{m}.$$

Choose now λ so small that all the cubes $\overline{C}_k$, $1 \le k \le k_{m+1}$, admit an embedding of a disjoint union $\coprod_{j=1}^{n_k} \overline{C}(\lambda)$ such that $n_k|C(\lambda)| > |C_k| - \epsilon/k_{m+1}$. In this way, we get an embedding of $\sum_{k=1}^{k_{m+1}} n_k$ closed cubes into M filling more than $\operatorname{Vol}(M) - 2\epsilon$.

If M is not compact, choose a volume-preserving embedding φ: $\overline{B^{2n}}(\operatorname{Vol}(M) - \epsilon) \hookrightarrow M$ (cf. Theorem 4.2) and apply the already proved part to $(\overline{B^{2n}}(\operatorname{Vol}(M) - \epsilon), \varphi^*\omega)$. $\square$

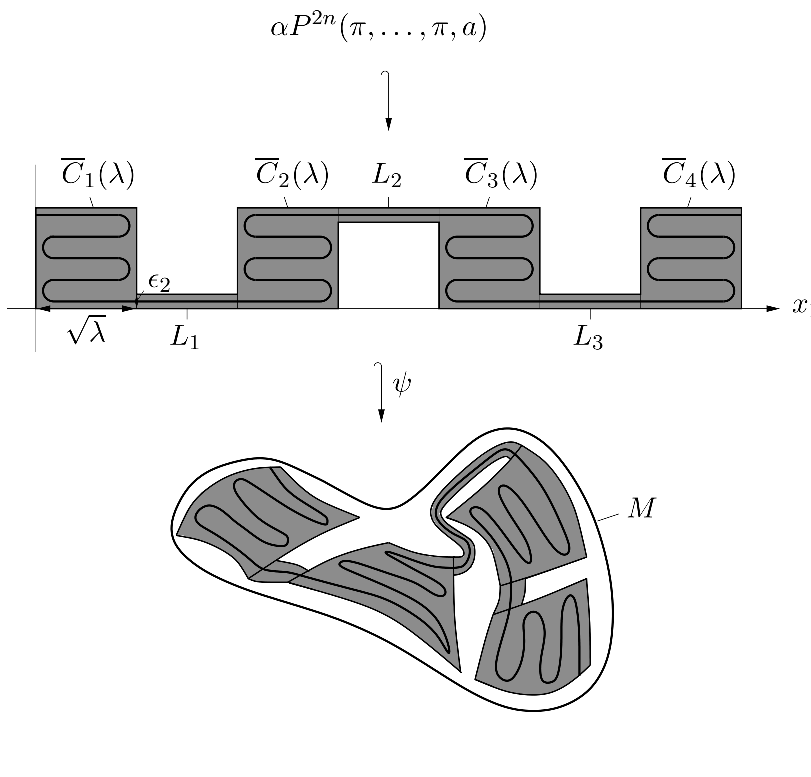

Figure 31: Asymptotic filling by polydiscs

We next connect the cubes by thin lines.

Pick $\epsilon_1 > 0$ and let $\varphi = \coprod_{i=1}^k \varphi_i \colon \coprod_{i=1}^k \overline{C}_i(\lambda) \hookrightarrow M$ be a corresponding embedding guaranteed by Lemma 4.7. Extensions of the φ_i to small neighbourhoods of $\overline{C}_i(\lambda)$ are still denoted by φ_i. We may assume that the faces of the $\overline{C}_i(\lambda)$ are cubes and that all the $\overline{C}_i(\lambda)$ lie in the positive cone of $\mathbb{R}^{2n}$ and touch the x_1-axis. Join these cubes by straight lines L_i as described in Figure 31, i.e. fixing regular parameterizations $L_i(t) \colon [0,1] \to L_i$ we have $L_i(0) \in \partial\overline{C}_i(\lambda),\ L_i(1) \in \partial\overline{C}_{i+1}(\lambda)$ and

$$L_i(t) = \begin{cases} (x_1(L_i(t)), 0, \dots, 0) & \text{for } i \text{ odd,} \\ (x_1(L_i(t)), \sqrt{\lambda}, \dots, \sqrt{\lambda}) & \text{for } i \text{ even.} \end{cases}$$

Let now $\coprod_{i=1}^{k-1} \lambda_i \colon \coprod L_i \to M \setminus \coprod \varphi_i(C_i(\lambda))$ be a disjoint family of embedded curves in M which touches $\coprod \varphi_i(\overline{C}_i(\lambda)$ only at the points $\lambda_i(0)$ and $\lambda_i(1)$ and coincides with $\varphi_{i|L_i}$ respectively $\varphi_{i+1|L_{i+1}}$ on a small neighbourhood of $\overline{C}_i(\lambda)$ respectively $\overline{C}_{i+1}(\lambda)$. Choose 1-parameter families of symplectic frames $\{e_{j,i}(t)\}_{j=1}^{2n}$ respectively $\{e'_{j,i}(t)\}_{j=1}^{2n}$ along $L_i(t)$ respectively $\lambda_i(L_i(t))$ such that $e_{1,i}(t) = \frac{d}{dt}\lambda_i(t)$ and $e'_{1,i}(t) = \frac{d}{dt}\lambda_i(L_i(t))$. Let $\tilde{\psi}_i$ be an extension of λ_i to a neighbourhood of L_i which coincides with φ_i respectively φ_{i+1} on a neighbourhood of $\lambda_i(0)$ respectively $\lambda_i(1)$ and which sends the symplectic frame along $L_i(t)$ to the one along $\lambda_i(L_i(t))$, i.e.

$$\left(T_{L_i(t)}\tilde{\psi}_i\right)(e_{j,i}(t)) = e'_{j,i}(t).$$

$\tilde{\psi}_i$ is thus a diffeomorphism on a neighbourhood of L_i which is symplectic along L_i. Using a variant of Mosers's method (see [26, Lemma 3.14 and its proof]) we see that $\tilde{\psi}_i$ may be deformed to an embedding ψ_i of a possibly smaller neighbourhood of L_i which still coincides with λ_i on L_i and φ_i respectively φ_{i+1} on a neighbourhood of $L_i(0)$ respectively $L_i(1)$, but is symplectic everywhere. Choose $\epsilon_2 > 0$ so small that for all i, ψ_i is defined on $N_i(\epsilon_2) = \{x_1(L_i(t))\} \times [0, \epsilon_2]^{2n-1}$ if i is odd and on $N_i(\epsilon_2) = \{x_1(L_i(t))\} \times [\sqrt{\lambda} - \epsilon_2, \sqrt{\lambda}]^{2n-1}$ if i is even.

Summing up, we see that there exists $\epsilon_2 > 0$ such that

$$\mathcal{N}(\epsilon_2) = \coprod C_i(\lambda) \coprod N_i(\epsilon_2)$$

symplectically embeds in M.

It remains to show that $\mathcal{N}(\epsilon_2)$ may be asymptotically filled by skinny polydiscs. We try to fill $\mathcal{N}(\epsilon_2)$ by $\alpha P^{2n}(\pi, \dots, \pi, a)$ with α small and a large by packing the $C_i(\lambda)$ as described in subsection 3.3.1 and using $N_i(\epsilon_2)$ to pass from $C_i(\lambda)$ to $C_{i+1}(\lambda)$. Here we think of $\alpha P^{2n}(\pi, \dots, \pi, a)$ as

$\frac{\alpha^2}{\epsilon_2}\square(a,\pi,\dots,\pi)\times\square(\epsilon_2,\dots,\epsilon_2)$ and of $C^{2n}(\lambda)$ as $\frac{1}{\epsilon_2}\square(\lambda,\dots,\lambda)\times\square(\epsilon_2,\dots,\epsilon_2)$. Write P_i for the restriction of the image of $\alpha P^{2n}(\pi,\dots,\pi,a)$ to $C_i(\lambda)$. In order to guarantee that the "right" face of P_i and the "left" face of $N_i(\epsilon_2)$ fit, we require that the number of folds in each z_1-z_2-layer is even and that the component of P_i between its right face and the last stairs touches $\partial\overline{C}_i(\lambda)$ wherever possible. This second point may be achieved by making $n-1$ of the stairs in P_i a little bit higher than necessary. The part of the image of $\alpha P^{2n}(\pi,\dots,\pi,a)$ between P_i and P_{i+1} will thus be contained in $N_i(\epsilon_2)$ whenever $\alpha^2\pi<\epsilon_2^2$.

Now, in Proposition 3.12 we have

$$\lim_{a\to\infty}\frac{a\pi^{n-1}}{(s^{2n}_{PC}(a))^n}=1,$$

and hence, by duality,

$$\lim_{q\to\infty}p^P_{1\dots1q}(C^{2n}(\lambda))=1. \tag{119}$$

(119) is clearly not affected by the two minor modifications which we required above for the packing of $C_i(\lambda)$. Thus half of the theorem follows.

As explained above, in order to prove the statement for ellipsoids we need the following non-elementary result.

Proposition 4.8 (McDuff-Polterovich, [25, Corollary 1.5.F]) *For each positive integer k, arbitrarily much of the volume of $\overline{C^{2n}}(\pi)$ may be filled by $n!\,k^n$ equal closed balls.*

This proposition may be proved in two different ways, either via symplectic blowing up and fibrations or via symplectic branched coverings. Combining it with Lemma 4.7, we see that we may fill as much of the volume of (M,ω) by disjoint equal closed balls as we want.

So assume that (M,ω) is almost filled by $m+1$ disjoint equal closed balls $\big(\overline{B}_i(\lambda),\varphi_i\big)$, $0\le i\le m$. By Lemma 3.11(ii) we may think of $B_i(\lambda)$ as fibered over $]i\lambda+i,(i+1)\lambda+i[\times]0,1[$ with fibers $\gamma\triangle^{n-1}(\lambda)\times\square^{n-1}(1)$, $1\ge\gamma>0$ (cf. Figure 32). Exactly as in the case of cubes we find an $\epsilon>0$ such that $\varphi=\coprod_{i=0}^m\varphi_i$ extends to a symplectic embedding ψ of a small neighbourhood of

$$\mathcal{N}(\epsilon)=\coprod B_i(\lambda)\bigcup\,]0,m\lambda+m[\times]0,\epsilon[^{2n-1}.$$

Let $\tau_i\colon\mathbb{R}^{2n}\to\mathbb{R}^{2n}$, $z\mapsto z+i(\epsilon-1,0,\dots,0)$ and set

$$\widetilde{N}_i(\epsilon)=\,]i\lambda+(i-2)\epsilon,i\lambda+i\epsilon[\times]0,1[\times]0,\epsilon[^{2n-2}$$

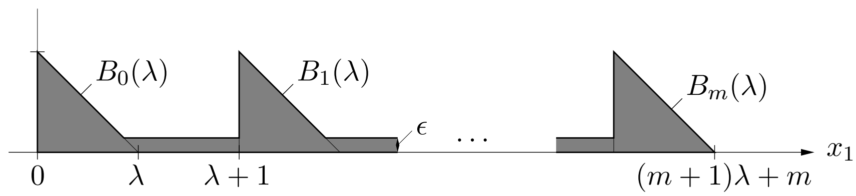

Figure 32: $\mathcal{N}(\epsilon)$

and

$$\widetilde{\mathcal{N}}(\epsilon) = \coprod_{i=0}^{m} \tau_i(B_i(\lambda)) \bigcup \coprod_{i=1}^{m} \widetilde{N}_i(\epsilon).$$

It is a simple matter to find a symplectomorphism σ of $\mathbb{R}^2$ such that $\sigma \times id_{2n-2}$ embeds $\widetilde{\mathcal{N}}(\epsilon)$ into an arbitrarily small neighbourhood of $\mathcal{N}(\epsilon)$. It thus remains to show that $\widetilde{\mathcal{N}}(\epsilon)$ may be asymptotically filled by skinny ellipsoids. We try to fill $\widetilde{\mathcal{N}}(\epsilon)$ by $\alpha E^{2n}(\pi,\dots,\pi,a)$ with α small and a large by packing the $B_i(\lambda)$ as in the proof of Proposition 3.13 and using $\widetilde{N}_i(\epsilon)$ to pass from $B_i(\lambda)$ to $B_{i+1}(\lambda)$. To this end, think of $\alpha E^{2n}(\pi,\dots,\pi,a)$ as fibered over $\square(\alpha^2 a, 1)$ with fibers $\frac{\beta^2}{\epsilon}\triangle^{n-1}(\pi)\times\square^{n-1}(\epsilon)$, $\alpha \geq \beta > 0$.

We observe that the present packing problem is easier then the one treated in Proposition 3.13 inasmuch as now only a part of $\alpha E^{2n}(\pi,\dots,\pi,a)$ is embedded into a $B_i(\lambda)$, whence the ellipsoid fibres decrease slowlier.

Let $\coprod_{i=1}^{l} P_i$ be a partition of $\tau_1(B_1(\lambda))$ as in the proof of Proposition 3.13 and let $\gamma\triangle^{n-1}(\lambda) \times \square^{n-1}(1)$ be the smallest fiber of P_{l-1}. Assume that l is so large that $\gamma\lambda < \epsilon$ and that α is so small that $\alpha^2\pi < \epsilon\gamma\lambda$. The image of the last ellipsoid fiber mapped to P_{l-1} is then contained in $\widetilde{N}_1(\epsilon)$, and we may pass to $\tau_2(B_2(\lambda))$. Having reached $P_1(\tau_2(B_2(\lambda)))$, we first of all move the ellipsoid fiber out of the connecting floor and then deform the fiber of the second floor to a fiber with maximal $\triangle^{n-1}$-factor (μ_1 in Figure 33). We then fill the remaining room in $P_1(\tau_2(B_2(\lambda)))$ as well as possible (cf. Figure 33) and proceed filling $\tau_2(B_2(\lambda))$ as before. The above modification in the filling of $\tau_2(B_2(\lambda))$ clearly does not affect the result in Proposition 3.13. Going on in the same way, we fill almost all of $\widetilde{\mathcal{N}}(\epsilon)$. This concludes the proof of Theorem 4.6. $\square$

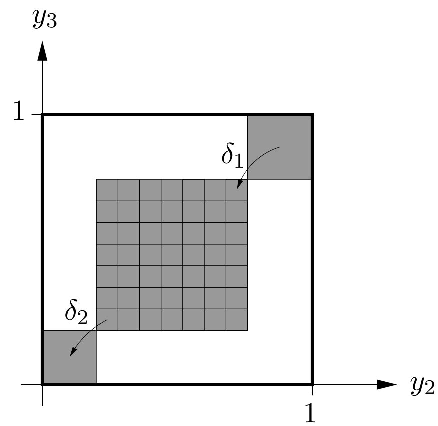

Figure 33: The two deformations in $P_1(\tau_2(B_2(\lambda)))$

4.2 Refined asymptotic invariants

Theorem 4.6 shows that the asymptotic packing numbers p_∞^E and p_∞^P are uninteresting invariants. However, we may try to recapture some symplectic information on the target space by looking at the convergence speed. Given (M, ω) with $\mathrm{Vol}\,(M, \omega) < \infty$ consider the function

$$[1, \infty[\to \mathbb{R}, \quad q \mapsto 1 - p_{1\dots 1q}^E(M, \omega)$$

and define a refined asymptotic invariant by

$$\alpha_E(M, \omega) = \sup\{\beta \,|\, 1 - p_{1\dots 1q}^E(M, \omega) = O(q^{-\beta})\}.$$

Define $\alpha_P(M, \omega)$ in a similar way.

Let $U \in \mathcal{O}(n)$ with piecewise smooth boundary ∂U. Given a subset $S \subset \partial U$, let

$$S_s = \{x \in U \,|\, d(x, S) < s\}$$

be the s-neighbourhood of S in U. We say that U is *admissible*, if there exists $\epsilon > 0$ such that $U \setminus \partial U_\epsilon$ is connected.

Example 4.9 Let $K(h, k) \subset \mathbb{R}^{2n}$ be a camel space:

$$K(h, k) = \{x_1 < 0\} \cup \{x_1 > 0\} \cup H(h, k),$$

where

$$H(h,k) = \left\{ \sum_{i=2}^n x_i^2 + \sum_{i=1}^n y_i^2 < h^2,\ x_1 = k \right\}.$$

Pick sequences $(h_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ with $h_1 > h_2 > \dots,\ h_i \to 0$ and $0 = k_1 < k_2 < \dots,\ k_i \to 1$, let $C = \{-1 < x_1, \dots, x_n, y_1, \dots, y_n < 1\}$ be a cube and set

$$U = C \cap \bigcap_{i=1}^{\infty} K(h_i, k_i).$$

Then C is not admissible. Thickening the walls and smoothing the boundary, we obtain non admissible sets with smooth boundaries. $\diamond$

Proposition 4.10 *Let $U \in \mathcal{O}(n)$ be admissible and let (M^{2n}, ω) be a closed symplectic manifold. Then*

$(i)_E \quad \alpha_E(U) \geq \frac{1}{n} \quad$ *if $n \leq 3$ or if $U \in \mathcal{E}(n)$*

$(ii)_E \quad \alpha_E(M,\omega) \geq \frac{1}{n} \quad$ *if $n \leq 3$*

$(i)_P \quad \alpha_P(U) \geq \frac{1}{n}$

$(ii)_P \quad \alpha_P(M,\omega) \geq \frac{1}{n}$.

Question. Given $\gamma \in]0, \frac{1}{2}[$, are there sets $U, V \in \mathcal{O}(2)$ with $\alpha_E(U) = \alpha_P(V) = \gamma$? Candidates for such necessarily non admissible sets are the sets described in Example 4.9 with $(h_i), (k_i)$ chosen appropriately. $\diamond$

Proof of Proposition 4.10. ad $(i)_P$. If U is a cube, the claim follows at once from Proposition 3.12. If U is an arbitrary admissible set, let

$$N_d = \{(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n} \,|\, x_i \in d\mathbb{Z},\ 1 \leq i \leq n\}$$

be the d-net in $\mathbb{R}^{2n}$, and let $\mathcal{C}_d$ be the union of all those open cubes in $\mathbb{R}^{2n} \setminus N_d$ which lie entirely in U. Observe that $U \setminus \mathcal{C}_d \subset \partial U_s\ \partial U_s$ whenever $d\sqrt{2n} < s$. Let $s_0 < \epsilon$ and $d_0 < s_0/\sqrt{2n}$. Pick α_0 much smaller than d_0 and exhaust $\mathcal{C}_{d_0}$ with $\frac{\alpha_0}{2} P^{2n}(\pi, \dots, \pi, a_0)$ by successively filling the cubes in

$\mathcal{C}_{d_0}$. More generally, let $k \in \mathbb{N}_0$, suppose that we almost exhausted $\mathcal{C}_{d_0/2^k}$ by $\frac{\alpha_0}{2^k} P^{2n}(\pi, \dots, \pi, a_k)$ and consider $\mathcal{C}_{d_0/2^{k+1}}$. Then

$$U \setminus \mathcal{C}_{d_0/2^{k+1}} \subset \partial U_{s_0/2^{k+1}}. \tag{120}$$

We fill the cubes in $\mathcal{C}_{d_0/2^k}$ by $\frac{\alpha_0}{2^{k+1}} P^{2n}(\pi, \dots, \pi, a_{k+1})$ in the same order as we filled them by $\frac{\alpha_0}{2^k} P^{2n}(\pi, \dots, \pi, a_k)$, but in between also fill the cubes in $\mathcal{C}_{d_0/2^{k+1}} \setminus \mathcal{C}_{d_0/2^k}$. Observe that in order to come back from a cube $C_{k+1} \in \mathcal{C}_{d_0/2^{k+1}}$ to its "mother-cube" $C_k \in \mathcal{C}_{d_0/2^k}$, we possibly have to use some extra space in C_k, but that for the subsequent filling by $\frac{\alpha_0}{2^{k+2}} P^{2n}(\pi, \dots, \pi, a_{k+2})$ this extra space will be halved.

Since the a_k were chosen maximal and since we exhaust more and more of U,

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = 2^n. \tag{121}$$

(121), the preceding remark and the case of a cube show that for any $\delta > 0$ there is a constant $C_1(\delta)$ such that for any k, any $k' \leq k$ and any $C_{k'} \in \mathcal{C}_{d_0/2^{k'}}$

$$\frac{\left| C_{k'} \setminus \operatorname{image}\left(\frac{\alpha_0}{2^{k'}} P^{2n}(\pi, \dots, \pi, a_k)\right) \right|}{|C_{k'}|} < C_1(\delta) a_k^{-\frac{1}{n}+\delta}. \tag{122}$$

Let $\partial_k U$ be the k-dimensional components of ∂U, $0 \leq k \leq 2n-1$, and let $|\partial_k U|$ be their k-dimensional volume. Then there are constants c_k depending only on U such that

$$\lim_{s \to 0^+} \frac{|\partial_k U_s|}{s^{2n-k}} = c_k,$$

whence

$$\lim_{s \to 0^+} \frac{\left|\partial U_{s/2}\right|}{|\partial U_s|} = \lim_{s \to 0^+} \frac{\left|\partial_{2n-1} U_{s/2}\right|}{|\partial_{2n-1} U_s|} = \frac{1}{2}. \tag{123}$$

(120), (123) and (122) imply that for any $\delta > 0$ there is a constant $C_2(\delta)$ such that for any k

$$\left| \mathcal{C}_{d_0/2^k} \setminus \operatorname{image}\left(\frac{\alpha_0}{2^k} P^{2n}(\pi, \dots, \pi, a_k)\right) \right| < C_2(\delta) a_k^{-\frac{1}{n}+\delta}. \tag{124}$$

Next, (120), (121) and (123) show that for any $\delta > 0$ there is a constant $C_3(\delta)$ such that for any k

$$|U \setminus \mathcal{C}_{d_0/2^k}| \leq |B_{s_0/2^k}| < C_3(\delta) a_k^{-\frac{1}{n}+\delta}. \tag{125}$$

(i)$_P$ now follows from (124) and (125).

ad (ii)$_P$. Cover M with Darboux charts (U_i, φ_i), $i = 1, \dots, m$, and choose admissible subsets V_i of U_i such that the sets $W_i = \varphi_i(V_i)$ are disjoint and $\bigcup_{i=1}^m \overline{W}_i = M$. Choose different points $p_i, q_i \in V_i$, set $\tilde{p}_i = \varphi_i(p_i)$, $\tilde{q}_i = \varphi_i(q_i)$, let $\widetilde{\lambda}_i \colon [0,1] \to M$ be a family of smooth, embedded and disjoint curves connecting $\tilde{q}_i$ with $\tilde{p}_{i+1}$, and set $\lambda_{i,j} = \varphi_j^{-1}(\widetilde{\lambda}_i)$, $1 \le i \le m-1$, $1 \le j \le m$. We may assume that near q_i respectively p_{i+1}, $\lambda_{i,i}$ respectively $\lambda_{i,i+1}$ are linear paths parallel to the x_1-axis. As in the proof of Theorem 4.6 we find $\epsilon > 0$ such that the $\widetilde{\lambda}_i$ extend to disjoint symplectic embeddings

$$\psi_i \colon [0,1] \times [-\epsilon, \epsilon]^{2n-1} \to M$$

whose compositions $\psi_{i,i} = \varphi_i^{-1} \circ \psi_i$ respectively $\psi_{i,i+1} = \varphi_{i+1} \circ \psi_i$ restrict to translations near $\{0\} \times [-\epsilon, \epsilon]^{2n-1}$ respectively $\{1\} \times [-\epsilon, \epsilon]^{2n-1}$. More generally, set $\psi_{i,i} = \varphi_j^{-1} \circ \psi_i$, and given $\delta \le \epsilon$, set

$$\psi_i^\delta = \psi_i|_{[0,1]\times[-\delta,\delta]^{2n-1}} \quad \text{ and } \quad \psi_{i,j}^\delta = \psi_{i,j}|_{[0,1]\times[-\delta,\delta]^{2n-1}}.$$

Let α be so small that $\alpha^2\pi < 4\delta^2$. We may then fill M with $\alpha P^{2n}(\pi, \dots, \pi, a)$ by successively filling $W_i \setminus \coprod_{k=1}^{m-1} \operatorname{image} \psi_k^\delta$ and passing from W_i to W_{i+1} with the help of ψ_i^δ.

In order to estimate the convergence speed of the filling of W_i, let us look at the corresponding filling of V_i instead. Set

$$\lambda_{i,j}^\delta = \{x \in V_i \mid d(x, \operatorname{image} \lambda_{i,j}) < \delta\} \quad \text{ and } \quad V_i^\delta = V_i \setminus \coprod_j \lambda_{i,j}^\delta.$$

Let L be a Lipschitz-constant for $\coprod_{i,j} \psi_{i,j}$. Then

$$\operatorname{image} \psi_{i,j}^\delta \;\subset\; \lambda_{i,j}^{L\delta}. \tag{126}$$

With V_i also V_i^0 is admissible, and so there is $\delta_0 > 0$ such that $V_i^{L\delta_0}$ is connected. This and (126) show that we may fill V_i with a part of $\alpha_0 P^{2n}(\pi, \dots, \pi, a_0)$ by entering V_i through $\lambda_{i,i}^{L\delta_0}$, filling as much of $V_i^{L\delta_0}$ as possible and leaving V_i through $\lambda_{i,i+1}^{L\delta_0}$. Let ${}_i\mathcal{C}_d^\delta$ be the union of those open cubes in $\mathbb{R}^{2n} \setminus N_d$ which lie entirely in $V_i^{L\delta}$. Then

$$V_i \setminus {}_i\mathcal{C}_{d_0}^{\delta_0} \;\subset\; \coprod_{j=1}^{m-1} \lambda_{i,j}^{2L\delta_0} \bigcup \, (\partial V_i)_{L\delta_0} \tag{127}$$

whenever $d_0\sqrt{2n} < L\delta_0$. Finally,

$$\lim_{s\to 0^+} \frac{\left|\lambda_{i,j}^{s/2}\right|}{\left|\lambda_{i,j}^{s}\right|} = \frac{1}{2^{2n-1}} < \frac{1}{2}. \tag{128}$$

(ii)$_P$ now follows from (127), (128) and the proof of (i)$_P$.

ad (i)$_E$ and (ii)$_E$. By the Folding Lemma, $E(\pi, a) \hookrightarrow P(\pi, (a+\pi)/2)$, whence the case $n = 2$ follows from (i)$_P$ and (ii)$_P$.

Let $n = 3$, and let U be a cube. We fill U as described in 3.3.2.1. This asymptotic packing problem resembles the one in the proof of Proposition 3.14. Again, for given a, the region in U not covered by the image of the maximal ellipsoid $\alpha E(\pi, \pi, a)$ fitting into U decomposes into several disjoint regions $R_h(a)$, $2 \leq h \leq 4$.

> $R_2(a)$ is the space needed for folding.
>
> $R_3(a)$ is the union of the space needed to deform the ellipsoid fibers and the space caused by the fact that the sum of the sizes of the ellipsoid fibres embedded into a column of the cube fibre and the x_3-width of the space needed to deform one of these ellipsoid fibres might be smaller than the size of the cube fibre.
>
> $R_4(a)$ is the space caused by the fact that the size of the ellipsoid fibres decreases during the filling of a column of the cube fibre.

We compare $R_h(a)$ with $R_h(2^n a) = R_h(8a)$. Let $\alpha' E(\pi, \pi, 8a)$ be the maximal ellipsoid fitting into U. A volume comparison shows that for a large α' is very close to $\alpha/2$. A similar but simpler analysis than in the proof of Proposition 3.14 now shows that given $\epsilon > 0$ there is a_0 such that for any $a \geq a_0$

$$(2-\epsilon)\left|R_h(8a)\right| < \left|R_h(a)\right|, \qquad 2 \leq h \leq 4.$$

This implies the claim in case of a cube. The general case follows from this case in the same way as (i)$_P$ and (ii)$_P$ followed from the case of a cube.

Finally, let $E = E(b_1, \dots, b_n)$. It follows from the description of Lagrangian folding in subsection 3.4 and from Lemma 3.15(i) that given $n-1$ relatively prime numbers $k_1, \dots, k_{n-1}$ there is an embedding $E^{2n}(\pi, \dots, \pi, a) \hookrightarrow \beta E(b_1, \dots, b_n)$ whenever

$$\left.\begin{array}{rcl} \frac{\pi}{\beta b_i} + \frac{\pi}{k_i \beta b_n} & < & \frac{1}{k_i}, \qquad 1 \leq i \leq n-1 \\ \frac{\pi}{\beta b_n} & < & \frac{k_1 \cdots k_{n-1}\pi}{a}. \end{array}\right\}. \tag{129}$$

W.l.o.g. we may set $b_n = 1$. (129) then reads

$$\left.\begin{array}{rcll} k_i\pi & < & (\beta-1)b_i, & 1 \le i \le n-1 \\ a & < & k_1 \cdots k_{n-1}\beta. & \end{array}\right\}. \tag{130}$$

Pick some (large) constant C and define β by

$$b_1 \cdots b_{n-1}\beta^n = \pi^{n-1}\left(a + Ca^{\frac{n-1}{n}}\right).$$

Moreover, pick $n-1$ prime numbers $p_1, \dots, p_{n-1}$, let l be the least common multiple of $\{p_i - p_j \,|\, 1 \le i < j \le n-1\}$, define m_i, $1 \le i \le n-1$, by

$$m_i = \max\{m \in \mathbb{N} \,|\, m_i l - p_i < (\beta-1)b_i/\pi\}$$

and set $k_i = m_i l - p_i$. We claim that the k_i are relatively prime. Indeed, assume that for some $i \neq j$

$$d \,|\, m_i l - p_i \quad \text{and} \quad d \,|\, m_j l - p_j. \tag{131}$$

Then d divides $(m_i l - p_i) - (m_j l - p_j) = p_i - p_j$, and hence, by the definition of l, d divides l. But then, by (131), d divides p_i and p_j, whence $d = 1$.

The first $n-1$ inequalities in (130) hold true by the definition of the k_i, and since $b_i \le 1$,

$$\begin{aligned} \pi^{n-1}k_1 \cdots k_{n-1}\beta &> (\beta b_1 - l - 1)\cdots(\beta_{n-1} - l - 1)\beta \\ &= b_1 \cdots b_{n-1}\beta^n + \sum_{i=1}^{n-1}(-1)^i c_i \beta^{n-i}, \end{aligned}$$

where the c_i are positive constants depending only on $b_1, \dots, b_{n-1}$ and l. For a large enough the last expression is larger than $b_1 \cdots b_{n-1}\beta^n - c_1\beta^{n-1}$, which equals

$$\pi^{n-1}\left(a + Ca^{\frac{n-1}{n}}\right) - c_1\left(\frac{\pi^{n-1}}{b_1 \cdots b_{n-1}}\right)^{\frac{n-1}{n}}\left(a + Ca^{\frac{n-1}{n}}\right)^{\frac{n-1}{n}}$$

and this is larger than $\pi^{n-1}a$ whenever a and C are large enough.

Finally, we have that

$$\frac{|E^{2n}(\pi, \dots, \pi, a)|}{|\beta E(b_1, \dots, b_n)|} = \frac{\pi^{n-1}a}{\beta^n b_1 \cdots b_{n-1}} = \frac{1}{1 + Ca^{-\frac{1}{n}}} = 1 - Ca^{-\frac{1}{n}} + o\left(a^{-\frac{1}{n}}\right),$$

from which the second claim in $(\mathrm{i})_E$ follows. $\square$

Remark. Suppose that we knew that there is a natural number k such that the cube C^{2n} admits a full symplectic packing by k equal balls and such that the space of symplectic embeddings of k equal balls into C^{2n} is unknotted. Combining such a result with Proposition 3.14 and the techniques used in the proof of Theorem 4.6 and Proposition 4.10 we may derive that

$$\alpha_E(U) \geq \frac{1}{2n} \quad \text{and} \quad \alpha_E(M,\omega) \geq \frac{1}{2n}$$

for any admissible $U \in \mathcal{O}(n)$ and any closed symplectic manifold (M^{2n},ω).

4.3 Higher order symplectic invariants

The construction of good higher order invariants for subsets of $\mathbb{R}^{2n}$ has turned out to be a difficult problem in symplectic topology. The known such invariants are Ekeland-Hofer capacities [6, 7] and symplectic homology [9, 10], which both rely on the variational study of periodic orbits of certain Hamiltonian systems, and the symplectic homology constructed via generating functions [32]. We propose here some higher order invariants which are based on an embedding approach.

Let (M^{2n},ω) be a symplectic manifold and let

$$e_1(M,\omega) = \sup\{A \,|\, B^{2n}(A) \text{ symplectically embeds in } (M,\omega)\}$$

be the Gromov-width of (M,ω). We inductively define $n-1$ other invariants by

$$e_i(M,\omega) = \sup\{A \mid E^{2n}(e_1(M,\omega),\dots,e_{i-1}(M,\omega),A,\dots,A) \\ \text{symplectically embeds in } (M,\omega)\}.$$

Similarly, given $U \in \mathcal{O}(n)$, let

$$e^n(U) = \inf\{A \,|\, U \text{ symplectically embeds in } B^{2n}(A)\}$$

and inductively define $n-1$ other invariants $e^i(U)$ by

$$e^i(U) = \inf\{A \,|\, U \text{ symplectically embeds in } E^{2n}(A,\dots,A,e^{i+1}(U),\dots,e^n(U)\}.$$

Clearly,

$$e_1(M,\omega) \leq e_2(M,\omega) \leq \cdots \leq e_n(M,\omega)$$

and

$$e^1(M,\omega) \leq e^2(M,\omega) \leq \cdots \leq e^n(M,\omega).$$

Moreover, $e_i(M, \alpha\omega) = |\alpha|\, e_i(M, \omega)$ and $e^i(U, \alpha\omega_0) = |\alpha|\, e^i(U, \omega_0)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$, and e_i and e^i are indeed invariants, that is $e_i(M, \omega) = e_i(N, \tau)$ and $e^i(U, \omega_0) = e^i(V, \omega_0)$ if there are symplectomorphisms $\varphi\colon (M, \omega) \to (N, \tau)$ and $\psi\colon (U, \omega_0) \to (V, \omega_0)$.

Example 4.11 Ekeland-Hofer capacities show that

$$e_i(E(a_1, \dots, a_n)) = a_i, \qquad 1 \le i \le n,$$

and

$$e^i(E(a_1, \dots, a_n)) = a_i, \qquad 1 \le i \le n, \qquad \text{if } 2a_1 \ge a_n.$$

$\diamond$

e_1 and e^n are also monotone and nontrivial, and are hence symplectic capacities (see [14] for the axioms of a symplectic capacity). This, however, does not hold true for any of the higher invariants. Indeed, let $Z(\pi) = D(\pi) \times \mathbb{R}^{2n-2}$ be the standard symplectic cylinder. Then

$$e_i(Z(\pi)) = \infty \quad \text{for all } i \ge 2.$$

Moreover, Example 4.11 and Theorem 2A show that none of the e_i, $i \ge 2$, is monotone, and the same holds true for e^i, $i \le n-1$. For instance, set $U_\lambda = \frac{4}{3} E(\lambda^{-1}\pi, \lambda\pi)$ and $V = E(\pi, 2\pi)$. By Theorem 4.6, U_λ symplectically embeds in V and $e^2(U_\lambda)$ is near to $\frac{4}{3}\pi$ if λ is large. Then also $e^1(U_\lambda)$ is near to $\frac{4}{3}\pi$; but $e^1(V) = \pi$.

Similar invariants may be constructed by looking at polydiscs instead of ellipsoids.

These considerations indicate that it should be difficult to construct higher order symplectic capacities via an embedding approach.

5 Appendix

A. Computer programs

All the Mathematica programs of this appendix may be found under

ftp://ftp.math.ethz.ch/pub/papers/schlenk/folding.m

For convenience, in the programs (but not in the text) both the u-axis and the capacity-axis are rescaled by a factor $1/\pi$.

A1. The estimate s_{EB}

As said at the beginning of 3.2.3.1 we fix a and u_1 and try to embed $E(\pi, a)$ into $B^4(2\pi + (1 - 2\pi/a)u_1)$ by multiple folding. If this works, we set $A(a, u_1) = 2\pi + (1 - 2\pi/a)u_1$ and $A(a, u_1) = a$ otherwise.

```
A[a_, u1_] :=
  Block[{A=2+(1-2/a)u1},
     j  = 2;
     uj = (a+1)/(a-1)u1-a/(a-1);
     rj = a-u1-uj;
     lj = rj/a;
     While[True,
           Which[EvenQ[j],
                   If[rj <= uj,
                      Return[A],
                      If[uj <= 2lj,
                         Return[a],
                         j++;
                         uj = a/(a-2)(uj-2lj);
                         rj = rj-uj;
                         li = lj;
                         lj = rj/a
                        ]
                     ],
                 OddQ[j],
                   If[rj <= uj+li,
                      Return[A],
                      j++;
                      uj = (a+1)/(a-1)uj;
                      rj = rj-uj;
                      lj = rj/a
                     ]
                ]
            ]
         ]
```

This program just does what we proposed to do in 3.2.3.1 in order to decide if the embedding attempt associated with u_1 succeeds or fails. Note, however, that in the `Oddq[j]`-part, we did not check whether the upper left corner of F_{j+2} is contained in $T(A, A)$. However, this negligence does not

cause troubles, since if the left edge of F_{j+2} indeed exceeds $T(A,A)$, the embedding attempt will fail in the subsequent `EvenQ[j+1]`-part. In fact, that the left edge of F_{j+2} exceeds $T(A,A)$ means that $l_{j+1} > u_{j+1}$; hence $r_{j+1} > u_{j+1}$ (since otherwise the embedding attempt would have succeeded in the preceding `OddQ[j]`-part), but $u_{j+1} \leq 2l_{j+1}$.

Writing again u_0 for the minimal u_1 which leads to an embedding, $A(a,u_1)$ is equal to a for $u_1 < u_0$ and it is a linear increasing function for $u_1 \geq u_0$. Since, by (23), we may assume that $u_0 \leq a/2$, we have $A(a,u_0) \leq \pi + a/2 < a$, whence u_0 is found up to accuracy $acc/2$ by the following bisectional algorithm.

```
u0[a_, acc_] :=
  Block[{},
      b = a/(a+1);
      c = a/2;
      u1 = (b+c)/2;
      While[(c-b)/2 > acc/2,
        If[A[a,u1] < a, c=u1, b=u1];
        u1 = (b+c)/2
           ];
      Return[u1]
        ]
```

Here the choice $b = a\pi/(a+\pi)$ is also based on (23). Up to accuracy acc, the resulting estimate $s_{EB}(a)$ is given by

```
sEB[a_, acc_] := 2 + (1-2/a)u0[a,acc].
```

A2. The estimate s_{EC}

Given a and u_1, we first calculate the height of the image of the corresponding embedding. The following program is easily understood by looking at Figure 19.

```
h[a_, u1_] :=
  Block[{l1=1-u1/a},
      j  = 2;
      uj = (a+1)/(a-1)u1-a/(a-1);
      rj = a-u1-uj;
      lj = rj/a;
      hj = 2l1;
```

```
    While[rj > u1+l1 - lj,
          j++;
          uj = (a+1)/(a-1)uj;
          rj = rj-uj;
          li = lj;
          lj = rj/a;
          If[EvenQ[j], hj = hj+2li]
         ];
    Which[EvenQ[j],
          hj = hj+lj,
          OddQ[j],
          hj = hj+Max[li,2lj]
         ];
    Return[hj]
      ]
```

As explained in 3.2.4.1, the optimal folding point u_1 is the u-coordinate of the unique intersection point of $h(a, u_1)$ and $w(a, u_1)$. It may thus be found again by a bisectional algorithm.

```
u0[a_, acc_] :=
  Block[{},
    b = a/(a+1);
    c = a/2;
    u1 = (b+c)/2;
    While[(c-b)/2 > acc/2,
      If[h[a,u1] > 1+(1-1/a)u1, b=u1, c=u1];
      u1 = (b+c)/2
         ];
    Return[u1]
     ]
```

Again, the choices $b = a\pi/(a + \pi)$ and $c = a/2$ reflect that we fold at least twice in which case $u_1 \geq l_1$ must hold true. Up to accuracy acc, the resulting estimate $s_{EC}(a)$ is given by

```
sEC[a_, acc_] := 1+(1-1/a)u0[a,acc].
```

B. Report on the Gromov width of closed symplectic manifolds

Recall that given any symplectic manifold (M^{2n}, ω) its Gromov width is defined by

$$w(M, \omega) = \sup\{c \mid \text{there is a symplectic embedding } (B^{2n}(c), \omega_0) \hookrightarrow (M, \omega)\}.$$

Historically, the width provided the first example of a symplectic capacity. Giving the size of the largest Darboux chart of (M, ω), the width is always positive, and in the closed case it is finite. We now restrict to closed manifolds and define an equivalent packing invariant by

$$p(M^{2n}, \omega) = \frac{|B^{2n}(w(M, \omega))|}{\mathrm{Vol}(M, \omega)} = \frac{w(M, \omega)^n}{n! \, \mathrm{Vol}(M, \omega)}.$$

In two dimensions the width is the volume and $p = 1$ (see Theorem 4.2). The basic result to discover rigidity in higher dimensions is a version of Gromov's Non-Squeezing Theorem [22].

Non-Squeezing Theorem (compact version) *Let (M^{2n}, ω) be closed, let σ be an area form on S^2 such that $\int_{S^2} \sigma = 1$ and assume that there is a symplectic embedding $B^{2n+2}(c) \hookrightarrow (M \times S^2, \omega \oplus a\sigma)$. Then $a \geq c$.*

Remark. More generally, let $S^2 \hookrightarrow M \ltimes S^2 \xrightarrow{\pi} M$ be an oriented S^2-bundle over a closed manifold M and let ω be a symplectic form on $M \ltimes S^2$ whose restriction to the fibers is nondegenerate and induces the given orientation. In particular, $a = \langle [\omega], [pt \times S^2] \rangle > 0$. Then the proof of the above Non-Squeezing Theorem also implies that $c \leq a$ whenever $B^{2n+2}(c)$ symplectically embeds in $(M \ltimes S^2, \omega)$. We will verify this below in the case where M is 2-dimensional. $\diamond$

Since the theory of J-holomorphic curves works best in dimension four, the deepest results on the Gromov-width have been proved for 4-manifolds. Given a symplectic 4-manifold (M, ω), let c_1 be the first Chern class of (M, ω) with respect to the contractible set of almost complex structures compatible with ω. Let $\mathcal{C}$ be the class of symplectic 4-manifolds (M, ω) for which there exists a class $A \in H_2(M; \mathbb{Z})$ with non-zero Gromov invariant and $c_1(A) + A^2 \neq 0$. Recall that a symplectic 4-manifold is called *rational* if it is the symplectic blow-up of $\mathbb{CP}^2$ and that it is said to be *ruled* if it is an S^2-bundle over a Riemann surface. The class $\mathcal{C}$ consists of symplectic blow-ups of

- rational and ruled manifolds;
- manifolds with $b_1 = 0$ and $b_2^+ = 1$;
- manifolds with $b_1 = 2$ and $(H^1(M;\mathbb{Z}))^2 \neq 0$.

We refer to [24] for more information on the class $\mathcal{C}$.

Recall that by definition an *exceptional sphere* in a symplectic 4-manifold (M,ω) is a symplectically embedded 2-sphere S of self-intersection number $S \cdot S = -1$, and that (M,ω) is said to be *minimal* if it contains no exceptional spheres. Combining the technique of symplectic blowing-up with Taubes theory of Gromov invariants, Biran [2, Theorem 6.A] showed that for the symplectic 4-manifolds (M,ω) in class $\mathcal{C}$ all packing obstructions come from exceptional spheres in the symplectic blow-up of (M,ω) and from the volume constraint. His result suffices to compute the Gromov-width of all minimal manifolds in the class $\mathcal{C}$.

Theorem 5.1 (Biran [2, Theorem 2.F]) *Let (M,ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$ which is minimal and neither rational nor ruled. Then $p(M,\omega) = 1$.*

Examples of manifolds satisfying the conditions of the above theorem are hyper-elliptic surfaces and the surfaces of Barlow, Dolgachev and Enriques, all viewed as Kähler surfaces.

We next look at minimal manifolds which are rational or ruled.

Let ω_{SF} be the unique U(3)-invariant Kähler form on $\mathbb{CP}^2$ whose integral over $\mathbb{CP}^1$ equals π. In the rational case, by a theorem of Taubes [30], (M,ω) is symplectomorphic to $(\mathbb{CP}^2, a\omega_{SF})$ for some $a > 0$, thus $p(M,\omega) = 1$.

Denote by Σ_g the Riemann surface of genus g. There are exactly two orientable S^2-bundles with base Σ_g, namely the trivial bundle $\pi\colon \Sigma_g \times S^2 \to \Sigma_g$ and the nontrivial bundle $\pi\colon \Sigma_g \ltimes S^2 \to \Sigma_g$ [26, Lemma 6.25]. Such a manifold is called a ruled surface. $\Sigma_g \ltimes S^2$ is the projectivization $\mathbb{P}(L_1 \oplus \mathbb{C})$ of the complex rank two bundle $L_1 \oplus \mathbb{C}$ over Σ_g, where L_1 is a holomorphic line bundle of Chern index 1. A symplectic form ω on a ruled surface is called *compatible* with the given ruling π if it restricts on each fiber to a symplectic form. Such a symplectic manifold is then called a *ruled symplectic manifold*. It is known that every symplectic structure on a ruled surface is diffeomorphic to a form compatible with the given ruling π via a diffeomorphism which acts trivially on homology, and that two cohomologous symplectic forms compatible with the same ruling are isotopic [21]. A symplectic form

ω on a ruled surface is thus determined up to diffeomorphism by the class $[\omega] \in H^2(M;\mathbb{R})$.

Fix now an orientation of the fibers of the given ruled symplectic manifold. We say that ω is *admissible* if its restriction to each fiber induces the given orientation.

Consider first the trivial bundle $\Sigma_g \times S^2$ with its given orientation, and let $\{B = [\Sigma_g \times pt], F = [pt \times S^2]\}$ be a basis of $H^2(M;\mathbb{Z})$ (here and henceforth we identify homology and cohomology via Poincaré duality). Then a cohomology class $c = bB + aF$ can be represented by an admissible form if and only if $c(B) = a > 0$ and $c(F) = b > 0$. We write $\Sigma_g(a) \times S^2(b)$ for this ruled symplectic manifold.

In case of the nontrivial bundle $\Sigma_g \ltimes S^2$ a basis of $H^2(\Sigma_g \ltimes S^2;\mathbb{Z})$ is given by $\{A, F\}$, where A is the class of a section with selfintersection number -1 and F is the fiber class. Set $B = A + \dfrac{F}{2}$. $\{B, F\}$ is then a basis of $H^2(\Sigma_g \ltimes S^2;\mathbb{R})$ with $B \cdot B = F \cdot F = 0$ and $B \cdot F = 1$. It turns out that in case $g = 0$ a form $c = bB + aF$ can be represented by an admissible form if and only if $a > \frac{b}{2} > 0$, while in case $g \geq 1$ this is possible if and only if $a > 0$ and $b > 0$ [26, Theorem 6.27]. We write $(\Sigma_g \ltimes S^2, \omega_{ab})$ for this ruled symplectic manifold.

Finally note that each admissible form is cohomologous to a standard Kähler form. For the trivial bundles these are just the split forms, and for the non-trivial bundles we refer to [17, p. 276].

Theorem 5.2 *Let (M^4,ω) be a ruled symplectic manifold, i.e. either $(M,\omega) = \Sigma_g(a) \times S^2(b)$ or $(M,\omega) = (\Sigma_g \ltimes S^2, \omega_{ab})$. If $(M,\omega) = S^2(a) \times S^2(b)$ we may assume that $a \geq b$. Then*

(i) $p(S^2(a) \times S^2(b)) \;=\; p(S^2 \ltimes S^2, \omega_{ab}) \;=\; \frac{b}{2a}$

(ii) $p(\Sigma_g(a) \times S^2(b)) \;=\; p(\Sigma_g \ltimes S^2, \omega_{ab}) \;=\; \min\{1, \frac{b}{2a}\}$ *if* $g \geq 1$

The statements for the trivial bundles are proved in [2, Theorem 6.1.A], and the ones for the non-trivial bundles are calculated in [29]. Observe that the upper bounds predicted by the Non-Squeezing Theorem and the volume condition are sharp in all cases. Explicit maximal embeddings are easily found for $g = 0$ and for $g \geq 1$ if $a \geq b$ [29], but no explicit maximal embeddings are known for $g \geq 1$ if $a < b$.

Also notice that $p(S^2(b) \times \Sigma_g(a)) = \min\{1, \frac{b}{2a}\}$ if $g \geq 1$ implies that the Non-Squeezing Theorem does not remain valid if the sphere is replaced by any other closed surface.

If (M^4, ω) does not belong to the class $\mathcal{C}$ only very few is known about $p(M, \omega)$. Indeed, no obstructions to full packings are known. Some flexibility results for products of higher genus surfaces were found by Jiang.

Theorem 5.3 (Jiang [15, Corollary 3.3 and 3.4]) *Let Σ be any closed surface of area $a > 1$.*

(i) Let T^2 be the 2-torus. There is a constant $C > 0$ such that $p(T^2(1) \times \Sigma(a)) \geq C$.

(ii) Let $g \geq 2$. There is a constant $C(g) > 0$ depending only on g such that $w(\Sigma_g(1) \times \Sigma(a)) \geq C(g) \log a$.

Remark. If $\Sigma = S^2$ Birans sharp result in Theorem 5.2 is of course much better. $\diamond$

Example 5.4 Set $R(a) = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\ 0 < y < a\}$, and consider the linear symplectic map

$$\begin{aligned} \varphi\colon (R(a) \times R(a), dx_1 \wedge dy_1 + dx_2 \wedge dy_2) &\to (\mathbb{R}^2 \times \mathbb{R}^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\ (x_1, y_1, x_2, y_2) &\mapsto (x_1 + y_2, y_1, -y_2, y_1 + x_2). \end{aligned}$$

Let $p\colon \mathbb{R}^2 \to T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ be the projection onto the standard symplectic torus. Then $p \circ \varphi\colon R(a) \times R(a) \to T^2 \times \mathbb{R}^2$ is an embedding; indeed, given (x_1, y_1, x_2, y_2) and (x_1', y_1', x_2', y_2') with

$$x_1 + y_2 \equiv x_1' + y_2' \mod \mathbb{Z} \tag{132}$$
$$y_1 \equiv y_1' \mod \mathbb{Z} \tag{133}$$
$$-y_2 = -y_2' \tag{134}$$
$$y_1 + x_2 = y_1' + x_2' \tag{135}$$

(134) gives $y_2 = y_2'$ and thus (132) implies $x_1 \equiv x_1' \mod \mathbb{Z}$ whence $x_1 = x_1'$. Moreover, (133) and (135) show that $y_1 - y_1' = x_2' - x_2 \equiv 0 \mod \mathbb{Z}$, hence $x_2 = x_2'$ and $y_1 = y_1'$.

Next observe that $p \circ \varphi(R(a) \times R(a)) \subset T^2 \times]-a, 0[\times]-a-1, a+1[$. Thus $R(a) \times R(a)$ embeds in $T^2(1) \times \Sigma(2a(a+1))$, and since $B^4(a)$ embeds in $R(a) \times R(a)$ and $B^4(1)$ embeds in $T^2(1) \times \Sigma(a)$ for any $a \geq 1$, we have shown

Proposition 5.5 *Let $a \geq 1$. Then*

$$p(T^2(1) \times \Sigma(a)) \geq \frac{\max\{a + 1 - \sqrt{2a+1}, 2\}}{4a}.$$

In particular, the constant C in Theorem 5.3(i) can be chosen to be $C = 1/8$.
$\diamond$

It would be interesting to have a complete list of those symplectic 4-manifolds with $p(M,\omega) = 1$. As we have seen above, the minimal such manifolds in class $\mathcal{C}$ are those which are not ruled, the trivial bundles $\Sigma(a) \times S^2(b)$ with $g(\Sigma) \geq 1$ and $a \geq 2b$ and the nontrivial bundles $(\Sigma \ltimes S^2, \omega_{ab})$ with $g(\Sigma) \geq 1$ and $a \leq 0$. Combining the techniques of [2] with Donaldson's existence result for symplectic submanifolds, Biran [3] found examples with $p(M,\omega) = 1$ which do not belong to $\mathcal{C}$.

In higher dimensions almost no flexibility results are known. Note however that for the standard Kähler form ω_{SF} on $\mathbb{C}P^n$ we have $p(\mathbb{C}P^n, \omega_{SF}) = 1$ (see e.g. [25]), and that the technique used in Example 5.4 shows that given any constant form ω on T^{2n} and an area form σ on Σ with $\int_\Sigma \sigma = 1$ there is a constant $C > 0$ such that $p(T^{2n} \times \Sigma, \omega \oplus a\sigma) \geq C$ ([15, Theorem 3.1]).

References

[1] P. Biran. The Geometry of Symplectic Packing. Ph.D. thesis. Tel-Aviv University, 1997.

[2] P. Biran. Symplectic packing in dimension 4. *Geom. Funct. Anal.* **7(3)** (1997) 420-437.

[3] P. Biran. A stability property of symplectic packing. Preprint 1997.

[4] K. Cieliebak. Symplectic boundaries: creating and destroying closed characteristics. *Geom. Funct. Anal.* **7(2)** (1997) 269-321.

[5] M. Demazure. Surfaces de del Pezzo II-V. In: *Séminaire sur les Singularités des Surfaces* (1976 - 1977), Lect. Notes Math., vol 777, pp. 23-69, Springer 1980.

[6] I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics. *Math. Z.* **200** (1990) 355-378.

[7] I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics II. *Math. Z.* **203** (1990) 553-567.

[8] Y. Eliashberg and H. Hofer. Unseen symplectic boundaries. *Manifolds and geometry* (Pisa, 1993) 178-189. Sympos. Math. XXXVI. Cambridge Univ. Press 1996.

[9] A. Floer and H. Hofer. Symplectic Homology I: Open sets in $\mathbb{C}^n$. *Math. Z.* **215** (1994) 37-88.

[10] A. Floer, H. Hofer and K. Wysocki. Applications of symplectic homology I. *Math. Z.* **217** (1994) 577-606.

[11] R. Greene and K. Shiohama. Diffeomorphisms and volume preserving embeddings of non-compact manifolds. *Transactions of the American Mathematical Society* **255** (1979) 403-414.

[12] M. Gromov. Pseudo-holomorphic curves in symplectic manifolds. *Invent. math.* **82** (1985) 307-347.

[13] M. Gromov. *Partial Differential Relations*. Springer 1986.

[14] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser 1994.

[15] M.-Y. Jiang. Symplectic embeddings of $\mathbb{R}^{2n}$ into some manifolds. Preprint Peking University (1997).

[16] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Volume II, Interscience, New York 1969.

[17] F. Lalonde. Isotopy of symplectic balls, Gromov's radius and the structure of ruled symplectic 4-manifolds. *Math. Ann.* **300(2)** (1994) 273–296.

[18] F. Lalonde and D. Mc Duff. The geometry of symplectic energy. *Ann. of Math.* **141** (1995) 349-371.

[19] F. Lalonde and D. Mc Duff. Hofer's L^∞-geometry: energy and stability of Hamiltonian flows, part II. *Invent. math.* **122** (1995) 35-69.

[20] F. Lalonde and D. Mc Duff. Local non-squeezing theorems and stability. *Geom. Funct. Anal.* **5(2)** (1995) 365-386.

[21] F. Lalonde and D. Mc Duff. The classification of ruled symplectic 4-manifolds. *Math. Res. Lett.* **3(6)** (1996) 769–778.

[22] F. Lalonde, D. Mc Duff and L. Polterovich. In preparation.

[23] D. Mc Duff. Blowing up and symplectic embeddings in dimension 4. *Topology* **30(3)** (1991) 409-421.

[24] D. Mc Duff. From symplectic deformation to isotopy. *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), 85–99, First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 1998.

[25] D. Mc Duff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. math.* **115** (1994) 405-429.

[26] D. Mc Duff and D. Salamon. *Introduction to Symplectic Topology.* Oxford Mathematical Monographs, Clarendon Press 1995.

[27] K. Nomizu and H. Ozeki. The existence of complete Riemannian metrics. *Proc. Amer. Math. Soc.* **12** (1961) 889-891.

[28] V. Ozols. Largest normal neighborhoods. *Proc. Amer. Math. Soc.* **61** (1976) 99-101.

[29] F. Schlenk. Some new explicit packings of symplectic 4-manifolds. In preparation.

[30] C. Taubes. $SW \Rightarrow Gr$: from the Seiberg-Witten equations to pseudo-holomorphic curves. *J. Amer. Math. Soc.* **9(3)** (1996) 845-918.

[31] L. Traynor. Symplectic packing constructions. *J. Differential Geom.* **42** (1995) 411-429.

[32] L. Traynor. Symplectic homology via generating functions. *Geom. Funct. Anal.* **4(6)** (1994) 718-748.

[33] H. Whitney. Differentiable manifolds. *Ann. of Math.* **37** (1936) 645-680.

Felix Schlenk, Mathematik, ETH Zentrum, 8092 Zürich, Switzerland

E-mail address: felix@math.ethz.ch